

ON THE TENSOR PRODUCT OF MODULES OVER SKEW MONOIDAL CATEGORIES

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ABSTRACT. This paper is about skew monoidal tensored \mathcal{V} -categories (= skew monoidal hommed \mathcal{V} -categories) and their categories of modules. A module over $\langle \mathcal{M}, *, R \rangle$ is an algebra for the monad $T = R * _$ on \mathcal{M} . We study in detail the skew monoidal structure of \mathcal{M}^T and construct a skew monoidal forgetful functor $\mathcal{M}^T \rightarrow {}_E\mathcal{M}$ to the category of E -objects in \mathcal{M} where $E = \mathcal{M}(R, R)$ is the endomorphism monoid of the unit object R . Then we give conditions for the forgetful functor to be strong monoidal and for the category \mathcal{M}^T of modules to be monoidal. In formulating these conditions a notion of ‘self-cocomplete’ subcategories of presheaves appears to be useful which provides also some insight into the problem of monoidality of the skew monoidal structures found by Altenkirch, Chapman and Uustalu on functor categories $[\mathcal{C}, \mathcal{M}]$.

1. INTRODUCTION

A skew monoidal category consists of a category \mathcal{M} , a functor $\mathcal{M} \times \mathcal{M} \xrightarrow{*} \mathcal{M}$, an object $R \in \mathcal{M}$ and comparison morphisms $\gamma : L * (M * N) \rightarrow (L * M) * N$, $\eta : M \rightarrow R * M$, $\varepsilon : M * R \rightarrow M$ satisfying usual monoidal category axioms but without the assumption that they are isomorphisms. This structure, more primitive than a monoidal category, turned out to contain all algebraic information about a bialgebroid over a non-commutative ring R . As it was shown in [19, Theorem 9.1] the closed skew monoidal structures on $\mathbf{Mod}\text{-}R$ with unit object being the right regular R -module are precisely the right bialgebroids B over R . Under this correspondence the category of right B -modules becomes, not a ‘module category’ over the skew monoidal \mathcal{M} but simply, the Eilenberg-Moore category \mathcal{M}^T of the monad $T = R * _$ on \mathcal{M} . This canonical monad is present also in any monoidal category although it is an uninteresting identity monad. For skew monoidal categories, however, it is the structure of \mathcal{M}^T that embodies the representation theory of the skew monoidal \mathcal{M} if viewed as a generalized bialgebroid.

It is well-known that the category of modules over a bialgebra, weak bialgebra or bialgebroid B has a monoidal structure. It is defined using the coalgebra structure of B and the forgetful functor to the underlying monoidal category $\mathbf{Mod}\text{-}k$ or $R\text{-}\mathbf{Mod}\text{-}R$. In case of a skew monoidal \mathcal{M} there is no a priori given underlying monoidal category, only the category \mathcal{M} itself. This makes the construction of any (skew?) monoidal product on \mathcal{M}^T non-trivial and in general impossible unless certain right exactness conditions are fulfilled. In the first version of this paper (presented in a talk [20]) we have constructed such a skew tensor product provided \mathcal{M} had reflexive coequalizers and $*$ preserved them. Independently, S. Lack and R. Street have given in [14] the same definition of the tensor but noticed also that it exists if $*$ preserves reflexive coequalizers only in the second argument. We call the tensor product \otimes on \mathcal{M}^T the ‘horizontal tensor’ in the hope that the ‘vertical tensor’ of comodules will also be found, although the latter so far has resisted all attempts.

Having defined the skew monoidal structure $\langle \mathcal{M}^T, \otimes, \bar{R} \rangle$ on the modules over \mathcal{M} the next step is to construct a skew monoidal forgetful functor to some underlying skew monoidal category. The natural candidate for the underlying category is the category ${}_E\mathcal{M}$ of E -objects in \mathcal{M} where E denotes the endomorphism monoid of R . In case of an R -bialgebroid ${}_E\mathcal{M}$ is just the bimodule category $R\text{-}\mathbf{Mod}\text{-}R$. The skew monoidal product $\underline{\otimes}$ on ${}_E\mathcal{M}$ can be introduced in two equivalent

ways: Either as the horizontal tensor product of an even more primitive skew monoidal structure on \mathcal{M} (which uses only the object R but not $*$), or as the Altenkirch-Chapman-Uustalu construction [3] applied to ${}_E\mathcal{M} \equiv [E, \mathcal{M}]$ by considering E as a 1-object category. The justification for this choice of $\bar{\otimes}$ is that a monadic and skew monoidal functor $\mathcal{G} : \mathcal{M}^T \rightarrow {}_E\mathcal{M}$ can be constructed.

We do all these constructions in the framework of tensored \mathcal{V} -categories. Assuming the existence of tensors $V \odot M$ of $V \in \mathcal{V}$ and $M \in \mathcal{M}$ is a technical assumption which, by the equivalence [9, 10] between enriched categories [12] with tensors and actegories [17] with hom objects, allows us to work with ordinary functors and natural transformations. This does not mean that skew monoidal categories without tensors are uninteresting. Skew monoidal monoids [21] are good examples. Only in Section 8 we make an exception by studying general \mathcal{V} -categories \mathcal{M} in order to characterize the functor categories $[\mathcal{C}, \mathcal{M}]$ for which the skew monoidal structure of [3] is monoidal. Introducing *self-cocompleteness* we find that if \mathcal{M} is a self-cocomplete subcategory of presheaves over \mathcal{C} , or, in the case of $\mathcal{C} = E$, it is a self-cocomplete subcategory $\mathcal{W} \subseteq \mathcal{V}_E$ of right E -modules then $[\mathcal{C}, \mathcal{M}]$, respectively ${}_E\mathcal{M}$, is a monoidal category. Under appropriate right exactness conditions on $\langle \mathcal{M}, *, R \rangle$ this implies that $\langle \mathcal{M}^T, \bar{\otimes}, \bar{R} \rangle$ is also monoidal and the forgetful functor \mathcal{G} is strong monoidal.

2. PREPARATIONS

2.1. Actegories. Let $\langle \mathcal{V}, \odot, I, a, l, r \rangle$ be a monoidal category in which the orientation of the coherence isomorphisms

$$\begin{aligned} a_{U,V,W} : U \odot (V \odot W) &\xrightarrow{\sim} (U \odot V) \odot W \\ l_V : V &\xrightarrow{\sim} I \odot V \\ r_V : V \odot I &\xrightarrow{\sim} V \end{aligned}$$

are chosen according to the right skew monoidal convention [19]. Therefore the monoidal category axioms (the redundant set of 5 axioms [16]), when written without inverses of a , l or r , look precisely as the skew monoidal¹ category axioms.

Definition 2.1. A left actegory over \mathcal{V} consists of

- a category \mathcal{M} ,
- a functor ${}_-\odot_- : \mathcal{V} \times \mathcal{M} \rightarrow \mathcal{M}$
- and natural isomorphisms

$$\begin{aligned} a_{U,V,M} : U \odot (V \odot M) &\xrightarrow{\sim} (U \odot V) \odot M \\ l_M : M &\xrightarrow{\sim} I \odot M \end{aligned}$$

satisfying one pentagon and two triangle conditions

$$\begin{aligned} (1) \quad & (a_{U,V,W} \odot M) \circ a_{U,V \odot W, M} \circ (U \odot a_{V,W,M}) = a_{U \odot V, W, M} \circ a_{U,V, W \odot M} \\ (2) \quad & a_{I,V,M} \circ l_{V \odot M} = l_V \odot M \\ (3) \quad & (r_V \odot M) \circ a_{V,I,M} \circ (V \odot l_M) = V \odot M. \end{aligned}$$

For typographical reasons we are using the same symbol for the action of \mathcal{V} on \mathcal{M} and for the monoidal product of \mathcal{V} and also the a and l have ambiguous meanings. Notice also the different form of the two triangle equations which is the consequence of our “skew” convention of orienting left and right unitors. The advantage is that we can immediately say what a left skew actegory would be over a skew monoidal \mathcal{V} . It has the same structure and axioms as above, just the invertibility of all coherence isomorphisms are relaxed.

¹Throughout this paper we use the term ‘skew monoidal’ for ‘right skew monoidal’.

In fact we will need skew actegories in the sequel although not left but right ones. So we give here the right and lax version of Definition 2.1 (see also [15])

Definition 2.2. A right skew actegory over a skew monoidal \mathcal{V} consists of

- a category \mathcal{M} ,
- a functor ${}_-\odot_- : \mathcal{M} \times \mathcal{V} \rightarrow \mathcal{M}$
- and natural transformations

$$a_{M,U,V} : M \odot (U \odot V) \rightarrow (M \odot U) \odot V$$

$$r_M : M \odot I \rightarrow M$$

satisfying one pentagon and two triangle conditions

$$(4) \quad (a_{M,U,V} \odot W) \circ a_{M,U \odot V,W} \circ (M \odot a_{U,V,W}) = a_{M \odot U,V,W} \circ a_{M,U,V \odot W}$$

$$(5) \quad (r_M \odot V) \circ a_{M,I,V} \circ (M \odot l_V) = M \odot V$$

$$(6) \quad r_{M \odot V} \circ a_{M,V,I} = M \odot r_V.$$

All skew monoidal categories in this paper are right skew monoidal. Thus both left and right actegories are considered over the same type of skew monoidal categories. Whether left or right, the actegory axioms appear as subsets of the skew monoidal category axioms if we disregard the types (\mathcal{M} or \mathcal{V}) in the arguments of the natural transformations.

Example 2.3. Let $\langle \mathcal{M}, *, R, \gamma, \eta, \varepsilon \rangle$ be any skew monoidal category. Then the Eilenberg-Moore category \mathcal{M}^T of the canonical monad $T = \langle R * -, \mu, \eta \rangle$ is a right skew actegory over \mathcal{M} . As a matter of fact, for a T -algebra $\bar{M} = \langle M, \alpha \rangle$ and for an object N of \mathcal{M} we can define the T -algebra

$$(7) \quad \bar{M} \triangleleft N := \langle M * N, (\alpha * N) \circ \gamma_{R,M,N} \rangle.$$

The associator and the right unitor of \mathcal{M} can be lifted to \mathcal{M}^T as natural transformations

$$(8) \quad \gamma_{\bar{M},L,K} : \bar{M} \triangleleft (L * K) \rightarrow (\bar{M} \triangleleft L) \triangleleft K$$

$$(9) \quad \varepsilon_{\bar{M}} : \bar{M} \triangleleft R \rightarrow \bar{M}.$$

They satisfy coherence conditions (4), (5) and (6) that make \mathcal{M}^T a right \mathcal{M} -actegory in the skew sense.

Definition 2.4. A morphism $\mathcal{M} \rightarrow \mathcal{N}$ of left \mathcal{V} -actegories consists of

- a functor $F : \mathcal{M} \rightarrow \mathcal{N}$
- and a natural transformation $F_{V,M} : V \odot FM \rightarrow F(V \odot M)$, called the strength,

subject to the coherence conditions

$$(10) \quad F_{V \odot W,M} \circ a_{V,W,FM} = F_{a_{V,W,M}} \circ F_{V,W \odot M} \circ (V \odot F_{V,M})$$

$$(11) \quad F_{I,M} \circ l_{FM} = F l_M.$$

The morphism F is called *strong* when its strength is an isomorphism and *strict* when its strength is the identity.

Definition 2.5. Let $F, G : \mathcal{M} \rightarrow \mathcal{N}$ be morphisms of left \mathcal{V} -actegories. A transformation $F \rightarrow G$ of actegory morphisms is nothing but a natural transformation $\nu : F \rightarrow G$ satisfying

$$(12) \quad \nu_{V \odot M} \circ F_{V,M} = G_{V,M} \circ (V \odot \nu_M).$$

The left \mathcal{V} -actegories, their morphisms and transformations are the objects, the 1-cells and 2-cells of a 2-category $\mathcal{V}\text{-ACT}$. As an instance of doctrinal adjunction [13] a morphism $F : \mathcal{M} \rightarrow \mathcal{N}$ in $\mathcal{V}\text{-ACT}$ has a right adjoint if and only if it is strong and the ordinary functor $F : \mathcal{M} \rightarrow \mathcal{N}$ has a right adjoint.

In order to deal with (skew) monoidal products on \mathcal{V} -actegories as actegory morphisms the 2-category $\mathcal{V}\text{-ACT}$ needs some extra structure. In its stead we choose a multicategory strategy by introducing multivariable morphisms of actegories. From here on \mathcal{V} is symmetric monoidal.

Definition 2.6. For left \mathcal{V} -actegories $\mathcal{M}_1, \dots, \mathcal{M}_n$ and \mathcal{N} an n -variable morphism $\mathcal{M}_1 \times \dots \times \mathcal{M}_n \rightarrow \mathcal{N}$ of left \mathcal{V} -actegories is a functor $F : \mathcal{M}_1 \times \dots \times \mathcal{M}_n \rightarrow \mathcal{N}$ such that

- (i) for each index $1 \leq k \leq n$ there is a natural transformation

$$F_{V, \underline{M}}^{(k)} : V \odot F(\underline{M}) \rightarrow F(V \odot_k \underline{M}),$$

where $V \odot_k \underline{}$ denotes the action of V on the k -th argument, such that for each fixed M_i , $i \neq k$, the $F_{V, \underline{M}}^{(k)}$ is a strength for the k -th partial functor

$$F(M_1, \dots, M_{k-1}, \underline{}, M_{k+1}, \dots, M_n) : \mathcal{M}_k \rightarrow \mathcal{N};$$

- (ii) for each pair of indices $1 \leq j < k \leq n$ the strengths $F^{(j)}$ and $F^{(k)}$ are compatible in the sense of

$$\begin{aligned} F_{V, U \odot_j \underline{M}}^{(k)} \circ (V \odot F_{U, \underline{M}}^{(j)}) &= F_{U, V \odot_k \underline{M}}^{(j)} \circ (U \odot F_{V, \underline{M}}^{(k)}) \circ \\ &\circ a_{U, V, F\underline{M}}^{-1} \circ (s_{V, U} \odot F\underline{M}) \circ a_{V, U, F\underline{M}} \end{aligned}$$

where $s_{U, V} : U \odot V \xrightarrow{\sim} V \odot U$ is the symmetry in \mathcal{V} .

For a parallel pair of n -variable actegory morphisms $F, G : \mathcal{M}_1 \times \dots \times \mathcal{M}_n \rightarrow \mathcal{N}$ a transformation $\nu : F \rightarrow G$ is a natural transformation that satisfies (12) in each variable.

In rare cases one needs a more general notion of actegory morphism than that of Definition 2.4.

Definition 2.7. For left actegories ${}_{\mathcal{V}}\mathcal{M}$ and ${}_{\mathcal{W}}\mathcal{N}$ a morphism ${}_{\mathcal{V}}\mathcal{M} \rightarrow {}_{\mathcal{W}}\mathcal{N}$ is a pair ${}_F F'$ consisting of a monoidal functor $F : \mathcal{V} \rightarrow \mathcal{W}$ and of a functor $F' : \mathcal{M} \rightarrow \mathcal{N}$ equipped with strength

$$F'_{V, M} : FV \odot F'M \rightarrow F'(V \odot M)$$

subject to the coherence conditions

$$(13) \quad F'_{V_1 \odot V_2, M} \circ (F_{V_1, V_2} \odot F'M) \circ a_{FV_1, FV_2, F'M} = F'a_{V_1, V_2, M} \circ F'_{V_1, V_2 \odot M} \circ (FV_1 \odot F'_{V_2, M})$$

$$(14) \quad F'_{I, M} \circ (F_0 \odot F'M) \circ l_{F'M} = F'l_M.$$

In other words, F' is a morphism of $\mathcal{M} \rightarrow F^*\mathcal{N}$ in $\mathcal{V}\text{-ACT}$ where $F^*\mathcal{N}$ is obtained from \mathcal{N} by “restriction of scalars”.

For a parallel pair ${}_F F', {}_G G'$ of actegory morphisms ${}_{\mathcal{V}}\mathcal{M} \rightarrow {}_{\mathcal{W}}\mathcal{N}$ a transformation ${}_F F' \rightarrow {}_G G'$ is a pair ${}_{\nu} \nu'$ consisting of a monoidal natural transformation $\nu : F \rightarrow G$ and of a natural transformation $\nu' : F' \rightarrow G'$ satisfying

$$\nu'_{V \odot M} \circ F'_{V, M} = G'_{V, M} \circ (\nu_V \odot \nu'_M).$$

The actegories with these generalized morphisms and transformations constitute a 2-category ACT which contains, as sub-2-categories, both the 2-category of monoidal categories and the 2-category $\mathcal{V}\text{-ACT}$ for each monoidal \mathcal{V} .

2.2. Enriched categories. Heretofore the base category \mathcal{V} was an arbitrary symmetric monoidal category. For this Subsection and for the rest of the paper \mathcal{V} will be a symmetric closed monoidal category with all small limits and colimits. As for the \mathcal{V} -actegories we restrict attention to those left \mathcal{V} -actegories \mathcal{M}_0 for which the action, as a functor of the first argument, $-\odot M : \mathcal{V} \rightarrow \mathcal{M}_0$, has a right adjoint $\mathcal{M}(M, -)$. That such \mathcal{V} -actegories are equivalent to \mathcal{V} -enriched categories is in the folklore for a long time. A precise formulation of the equivalence has been given by Gordon and Power in [9] for the general case of action by enrichment over a bicategory. For a more recent account on the subject see [10].

Let us summarize briefly how the \mathcal{V} -category structure emerges from the actegory structure. First one introduces the \mathcal{V} -valued hom $\mathcal{M}(M, N)$ as the value of the right adjoint $\mathcal{M}(M, -)$ on N . Then the unit and counit of the adjunction

$$(15) \quad \mathcal{M}_0(V \odot M, N) \cong \mathcal{V}(V, \mathcal{M}(M, N))$$

denoted by

$$\begin{aligned} \text{coev}_{M,V} : V &\rightarrow \mathcal{M}(M, V \odot M) \\ \text{ev}_{M,N} : \mathcal{M}(M, N) \odot M &\rightarrow N \end{aligned}$$

can be used to construct the strength of $\mathcal{M}(M, -)$ by

$$(16) \quad \begin{aligned} \chi_{V,M,N} &:= \mathcal{M}(M, V \odot \text{ev}_{M,N}) \circ \mathcal{M}(M, a_{V, \mathcal{M}(M,N), M}^{-1}) \circ \text{coev}_{M, V \odot \mathcal{M}(M,N)} \\ &: V \odot \mathcal{M}(M, N) \rightarrow \mathcal{M}(M, V \odot N). \end{aligned}$$

Equipped with this strength the functor $\mathcal{M}(M, -)$ becomes a morphism of \mathcal{V} -actegories $\mathcal{M}_0 \rightarrow \mathcal{V}$ and ev_M and coev_M become transformations of actegory morphisms, i.e., they satisfy

$$(17) \quad V \odot \text{ev}_{M,N} = \text{ev}_{M, V \odot N} \circ (\chi_{V,M,N} \odot M) \circ a_{V, \mathcal{M}(M,N), M}$$

$$(18) \quad \text{coev}_{M, U \odot V} = \mathcal{M}(M, a_{U, V, M}) \circ \chi_{U, M, V \odot M} \circ (U \odot \text{coev}_{M, V}).$$

Using these identities and the coherence conditions for the strength χ it is not difficult to show that the \mathcal{V} -category composition and unit defined by

$$(19) \quad c_{L,M,N} := \mathcal{M}(L, \text{ev}_{M,N}) \circ \chi_{\mathcal{M}(M,N), L, M} : \mathcal{M}(M, N) \odot \mathcal{M}(L, M) \rightarrow \mathcal{M}(L, N)$$

$$(20) \quad i_M := \mathcal{M}(M, l_M^{-1}) \circ \text{coev}_{M, I} : I \rightarrow \mathcal{M}(M, M)$$

obey the axioms of a \mathcal{V} -enriched category. This \mathcal{V} -category \mathcal{M} is then automatically tensored since the adjunction (15) readily implies the isomorphism

$$\mathcal{M}(V \odot M, N) \cong [V, \mathcal{M}(M, N)],$$

\mathcal{V} -natural in N where $[-, -]$ denotes left internal hom in \mathcal{V} .

This correspondence between hommed \mathcal{V} -actegories and tensored \mathcal{V} -categories extends to the morphisms and to the transformations in the following way.

Proposition 2.8. *Given tensored \mathcal{V} -categories \mathcal{M} and \mathcal{N} and given an object map $M \mapsto FM$ there is a bijection between*

- *the ‘arrow map’ data $F_{M,M'} : \mathcal{M}(M, M') \rightarrow \mathcal{N}(FM, FM')$ of a \mathcal{V} -functor $F : \mathcal{M} \rightarrow \mathcal{N}$*
- *and the strength data $F_{V,M} : V \odot FM \rightarrow F(V \odot M)$ of an actegory morphism $F : \mathcal{M}_0 \rightarrow \mathcal{N}_0$*

given by the following relations:

$$(21) \quad F_{V,M} = \hat{F}_{M, V \odot M} \circ (\text{coev}_{M,V} \odot FM)$$

$$(22) \quad F_{M,M'} = \mathcal{N}(FM, \hat{F}_{M,M'}) \circ \text{coev}_{FM, \mathcal{M}(M,M')}$$

where $\hat{F}_{M,M'} : \mathcal{M}(M, M') \odot FM \rightarrow FM'$ is the diagonal arrow in the commuting square

$$(23) \quad \begin{array}{ccc} \mathcal{M}(M, M') \odot FM & \xrightarrow{F_{M,M'} \odot FM} & \mathcal{N}(FM, FM') \odot FM \\ F_{\mathcal{M}(M, M'), M} \downarrow & & \downarrow \text{ev}_{FM, FM'} \\ F(\mathcal{M}(M, M') \odot M) & \xrightarrow{F \text{ ev}_{M, M'}} & FM' . \end{array}$$

Given \mathcal{V} -functors $F, G : \mathcal{M} \rightarrow \mathcal{N}$ between tensored \mathcal{V} -categories a natural transformation $\nu : F \rightarrow G$ is \mathcal{V} -natural if and only if it is a transformation of actegory morphisms.

Applying this Proposition to the partial functors of a multivariable actegory morphism F we obtain \mathcal{V} -functors $F_k : \mathcal{M}_k \rightarrow \mathcal{N}$ indexed by variables $M_i \in \mathcal{M}_i$, $i \neq k$, such that each pair $\langle F_j, F_k \rangle$ obeys a compatibility condition of the form [8, (4.4)] and therefore determines a unique \mathcal{V} -functor $F : \mathcal{M}_1 \otimes \dots \otimes \mathcal{M}_n \rightarrow \mathcal{N}$ on the tensor product of \mathcal{V} -categories. Note, however, that $\mathcal{M}_1 \otimes \dots \otimes \mathcal{M}_n$ is not a tensored \mathcal{V} -category in a natural way so there is no strength for such a \mathcal{V} -functor. In its stead there are n pieces of strengths, one for each partial \mathcal{V} -functor.

2.3. Colimits. For a tensored \mathcal{V} -category \mathcal{M} the general notion of \mathcal{V} -colimit [12] reduces to computation of tensors and enriched coends: For \mathcal{V} -functors $F : \mathcal{D} \rightarrow \mathcal{M}$ and $G : \mathcal{D}^{\text{op}} \rightarrow \mathcal{V}$ the colimit of F weighted by G is the coend

$$G \star F = \int^D GD \odot FD .$$

But if we assume also that the \mathcal{V} -valued contravariant hom $\mathcal{M}(_, M) : \mathcal{M}_0^{\text{op}} \rightarrow \mathcal{V}$ transforms colimits in \mathcal{M}_0 to limits in \mathcal{V} then the above coend can be computed by an ordinary colimit in \mathcal{M}_0 ,

$$\bigsqcup_{D, D'} \mathcal{D}(D, D') \odot (GD' \odot FD) \rightrightarrows \bigsqcup_D GD \odot FD \longrightarrow G \star F .$$

Lemma 2.9. *Let \mathcal{M} be a tensored \mathcal{V} -category and \mathcal{D} an ordinary small category. Then $\mathcal{M}(_, M)$ transforms colimits in \mathcal{M}_0 of functors $\mathcal{D} \rightarrow \mathcal{M}_0$ to limits in \mathcal{V} for all $M \in \mathcal{M}_0$ if and only if $V \odot _ : \mathcal{M}_0 \rightarrow \mathcal{M}_0$ preserves colimits of functors $\mathcal{D} \rightarrow \mathcal{M}_0$ for all $V \in \mathcal{V}$.*

Proof. This follows from the ordinary adjunction $_ \odot M \dashv \mathcal{M}(M, _)$ and from the fact that the (set valued) hom-functors preserve (transform) and collectively reflect (co)limits. \square

Corollary 2.10. *Let \mathcal{M}_0 be a left \mathcal{V} actegory for which the functors $_ \odot M : \mathcal{V} \rightarrow \mathcal{M}_0$ have right adjoints for all $M \in \mathcal{M}_0$. Then the emerging tensored \mathcal{V} -category \mathcal{M} is cocomplete if and only if*

- the ordinary category \mathcal{M}_0 is cocomplete,
- the functors $V \odot _ : \mathcal{M}_0 \rightarrow \mathcal{M}_0$ preserve (ordinary) colimits.

Proof. This is just a translation to the actegory language of [5, Vol. 2, Theorem 6.6.14]. \square

3. SKEW MONOIDAL ACTEGORIES

Let $\langle \mathcal{V}, \odot, I, a, l, r, s \rangle$ be a symmetric closed monoidal category with all small limits and colimits.

Definition 3.1. A skew monoidal \mathcal{V} -actegory consists of

- a \mathcal{V} -actegory \mathcal{M}_0 with action $_ \odot _ : \mathcal{V} \times \mathcal{M}_0 \rightarrow \mathcal{M}_0$ and with coherence morphisms

$$\begin{aligned} a_{U, V, M} : U \odot (V \odot M) &\rightarrow (U \odot V) \odot M \\ l_M : M &\rightarrow I \odot M \end{aligned}$$

- a 2-variable morphism $- * - : \mathcal{M}_0 \times \mathcal{M}_0 \rightarrow \mathcal{M}_0$ of \mathcal{V} -actegories, the skew tensor, with strengths

$$\begin{aligned}\Gamma_{V,M,N} &: V \odot (M * N) \rightarrow (V \odot M) * N \\ \Gamma'_{V,M,N} &: V \odot (M * N) \rightarrow M * (V \odot N),\end{aligned}$$

- an object R of \mathcal{M}_0 , called the skew unit,
- and transformations of actegory morphisms

$$\begin{aligned}\gamma_{L,M,N} &: L * (M * N) \rightarrow (L * M) * N \\ \eta_M &: M \rightarrow R * M \\ \varepsilon_M &: M * R \rightarrow M.\end{aligned}$$

The transformations are required to satisfy the skew monoidality axioms

$$(24) \quad (\gamma_{K,L,M} * N) \circ \gamma_{K,L,M,N} \circ (K * \gamma_{L,M,N}) = \gamma_{K*L,M,N} \circ \gamma_{K,L,M*N}$$

$$(25) \quad \gamma_{R,M,N} \circ \eta_{M*N} = \eta_M * N$$

$$(26) \quad \varepsilon_{M*N} \circ \gamma_{M,N,R} = M * \varepsilon_N$$

$$(27) \quad (\varepsilon_M * N) \circ \gamma_{M,R,N} \circ (M * \eta_N) = M * N$$

$$(28) \quad \varepsilon_R \circ \eta_R = R$$

Unpacking the definition we obtain, beyond the 3 actegory axioms and the 5 skew monoidality axioms, 10 more:

$$(29) \quad \Gamma_{U \odot V, M, N} \circ a_{U, V, M*N} = (a_{U, V, M} * N) \circ \Gamma_{U, V \odot M, N} \circ (U \odot \Gamma_{V, M, N})$$

$$(30) \quad \Gamma_{I, M, N} \circ l_{M*N} = l_M * N$$

$$(31) \quad \Gamma'_{U \odot V, M, N} \circ a_{U, V, M*N} = (M * a_{U, V, N}) \circ \Gamma'_{U, M, V \odot N} \circ (U \odot \Gamma'_{V, M, N})$$

$$(32) \quad \Gamma'_{I, M, N} \circ l_{M*N} = M * l_N$$

$$(33) \quad \begin{aligned} \Gamma'_{U, V \odot M, N} \circ (U \odot \Gamma_{V, M, N}) &= \Gamma_{V, M, U \odot N} \circ (V \odot \Gamma'_{U, M, N}) \circ \\ &\circ a_{V, U, M*N}^{-1} \circ (s_{U, V} \odot (M * N)) \circ a_{U, V, M*N} \end{aligned}$$

$$(34) \quad \gamma_{V \odot L, M, N} \circ \Gamma_{V, L, M*N} = (\Gamma_{V, L, M} * N) \circ \Gamma_{V, L*M, N} \circ (V \odot \gamma_{L, M, N})$$

$$(35) \quad \gamma_{L, V \odot M, N} \circ (L * \Gamma_{V, M, N}) \circ \Gamma'_{V, L, M*N} = (\Gamma'_{V, L, M} * N) \circ \Gamma_{V, L*M, N} \circ (V \odot \gamma_{L, M, N})$$

$$(36) \quad \gamma_{L, M, V \odot N} \circ (L * \Gamma'_{V, M, N}) \circ \Gamma'_{V, L, M*N} = \Gamma'_{V, L*M, N} \circ (V \odot \gamma_{L, M, N})$$

$$(37) \quad \Gamma'_{V, R, M} \circ (V \odot \eta_M) = \eta_{V \odot M}$$

$$(38) \quad \varepsilon_{V \odot M} \circ \Gamma_{V, M, R} = V \odot \varepsilon_M.$$

The first five are the coherence conditions for the 2-variable actegory morphism $\langle *, \Gamma, \Gamma' \rangle$ and the other five are the conditions for γ , η and ε to be transformations of \mathcal{V} -actegory morphisms.

Of course, the above Definition is too general for our purposes. We are interested in \mathcal{V} -actegories the underlying category of which is a tensored \mathcal{V} -category and which satisfies also some right exactness conditions.

Definition 3.2. Let \mathcal{V} be a symmetric monoidal closed category with small limits and colimits. A skew monoidal \mathcal{V} -actegory $\langle \mathcal{M}_0, \odot, a, l, *, \Gamma, \Gamma', R, \gamma, \eta, \varepsilon \rangle$ is called a *skew monoidal tensored \mathcal{V} -category* or, what is the same, a *skew monoidal hommed \mathcal{V} -actegory* if the following condition holds:

- (i) For each object M of \mathcal{M}_0 the functor $- \odot M : \mathcal{V} \rightarrow \mathcal{M}_0$ has a right adjoint $\mathcal{M}(M, -)$.

A skew monoidal tensored \mathcal{V} -category is called *r2-exact* if:

- (ii) The category \mathcal{M}_0 has reflexive coequalizers.
- (iii) For each object V of \mathcal{V} the functor $V \odot _ : \mathcal{M}_0 \rightarrow \mathcal{M}_0$ preserves reflexive coequalizers.
- (iv) For each object M of \mathcal{M}_0 the functor $M * _ : \mathcal{M}_0 \rightarrow \mathcal{M}_0$ is strong, i.e., Γ' is an isomorphism.
- (v) For each object M of \mathcal{M}_0 the functor $M * _ : \mathcal{M}_0 \rightarrow \mathcal{M}_0$ preserves reflexive coequalizers.

Similarly, we define *r1-exactness* by requiring, instead of (iv) and (v), invertibility of Γ and preservation of reflexive coequalizers by the functors $_ * M$. If \mathcal{M} is both r1-exact and r2-exact then it is called *r-exact*.

Lemma 3.3. *Let \mathcal{M} be a r2-exact skew monoidal \mathcal{V} -category. Then the category \mathcal{M}^T of modules over \mathcal{M} is a tensored \mathcal{V} category, has reflexive coequalizers and the functors $V \odot _ : \mathcal{M}^T \rightarrow \mathcal{M}^T$ preserve them.*

Proof. The underlying endofunctor of T , $R * _ : \mathcal{M}_0 \rightarrow \mathcal{M}_0$ is a strong morphism of actegories since $\Gamma'_{_, R, _}$ is invertible. The monad multiplication $\mu_M = (\varepsilon_R * M) \circ \gamma_{R, R, M}$ and the unit η_M are built from 2-cells of \mathcal{V} -ACT using composition and $*$, so they are 2-cells, too. The Eilenberg-Moore category of T carries a \mathcal{V} -category structure such that the forgetful functor $G : \mathcal{M}^T \rightarrow \mathcal{M}$ is a strict morphism of actegories. For a T -algebra $\bar{M} = \langle M, \alpha \rangle$ the action of $V \in \mathcal{V}$ has the form

$$V \odot \bar{M} = \left\langle V \odot M, T(V \odot M) \xrightarrow{\Gamma'_{V, R, M}^{-1}} V \odot TM \xrightarrow{V \odot \alpha} V \odot M \right\rangle.$$

The right adjoint of $_ \odot \bar{M}$ is just the equalizer in \mathcal{V}

$$\mathcal{M}^T(\bar{M}, \bar{N}) \longrightarrow \mathcal{M}(M, N) \xrightleftharpoons[\mathcal{M}(\alpha, N)]{\mathcal{M}(TM, \beta) \circ T_{M, N}} \mathcal{M}(TM, N)$$

for T -algebras $\bar{M} = \langle M, \alpha \rangle$, $\bar{N} = \langle N, \beta \rangle$. \mathcal{M}^T has reflexive coequalizers since T preserves and therefore the forgetful functor G creates them. Finally, $V \odot _$ preserves reflexive coequalizers in \mathcal{M} and commutes with T via Γ' , therefore using preservation and creation of reflexive coequalizers by G we conclude that $V \odot _$ preserves reflexive coequalizers in \mathcal{M}^T . \square

The above Lemma provides 3 of the 5 conditions for \mathcal{M}^T to be also a r2-exact skew monoidal \mathcal{V} -category. Of course, the last 2 conditions of Definition 3.2 are so far meaningless since a skew monoidal product on \mathcal{M}^T is not yet defined. The next Section provides the missing skew tensor.

4. SKEW MONOIDAL STRUCTURE ON \mathcal{M}^T

First we consider (ordinary) skew monoidal categories, forgetting the \mathcal{V} -structure for a while. Let $\langle \mathcal{M}, *, R, \gamma, \eta, \varepsilon \rangle$ be a skew monoidal category with coequalizers of reflexive pairs and with $*$ preserving these coequalizers in the 2nd argument. This implies that \mathcal{M}^T also has reflexive coequalizers and the forgetful functor $G : \mathcal{M}^T \rightarrow \mathcal{M}$ preserves them.

Let $\mu_{N, M} := (\varepsilon_N * M) \circ \gamma_{N, R, M}$ as in [19] which extends the monad multiplication: $\mu_M = \mu_{R, M}$. For each object N of \mathcal{M} and for each T -algebra $\bar{M} = \langle M, \alpha \rangle$ we choose a coequalizer

$$(39) \quad N * TM \xrightleftharpoons[N * \alpha]{\mu_{N, M}} N * M \xrightarrow{\pi_{N, \bar{M}}} N \bar{\otimes} \bar{M}$$

and call its colimit object the horizontal tensor product of N and \bar{M} . Note that the parallel pair is reflexive: $N * \eta_M$ is a common section. After having defined $\bar{\otimes}$ on the objects we extend its definition to a functor $\mathcal{M} \times \mathcal{M}^T \rightarrow \mathcal{M}$ in such a way that π becomes a natural transformation $\pi_{N, \bar{M}} : N * G\bar{M} \rightarrow N \bar{\otimes} \bar{M}$.

If N is replaced by the underlying object of a T -algebra $\langle N, \beta \rangle$ then $N \bar{\otimes} \bar{M}$ becomes the underlying object of a T -algebra $\bar{N} \bar{\otimes} \bar{M}$ the action ψ of which is uniquely determined by the diagram

$$(40) \quad \begin{array}{ccccc} R * (N * TM) & \xrightleftharpoons[R * (N * \alpha)]{R * \mu_{N,M}} & R * (N * M) & \xrightarrow{R * \pi_{N,\bar{M}}} & R * (N \bar{\otimes} \bar{M}) \\ \downarrow (\beta * TM) \circ \gamma_{R,N, TM} & & \downarrow (\beta * M) \circ \gamma_{R,N,M} & & \downarrow \psi \\ N * TM & \xrightleftharpoons[N * \alpha]{\mu_{N,M}} & N * M & \xrightarrow{\pi_{N,\bar{M}}} & N \bar{\otimes} \bar{M} \end{array}$$

where on the vertical arrows one can recognize the T -algebra actions of $\bar{N} \triangleleft TM$ and $\bar{N} \triangleleft M$, as we defined them in Example 2.3. Therefore $\pi_{N,\bar{M}}$ can be lifted to \mathcal{M}^T as the coequalizer

$$(41) \quad \bar{N} \triangleleft TM \xrightleftharpoons[N * \alpha]{\mu_{N,M}} \bar{N} \triangleleft M \xrightarrow{\pi_{\bar{N},\bar{M}}} \bar{N} \bar{\otimes} \bar{M}$$

where the new π is related to the old one by

$$G\pi_{\bar{N},\bar{M}} = \pi_{G\bar{N},\bar{M}}.$$

Lemma 4.1. *For any N and for any free T -algebra $\langle TM, \mu_M \rangle$ the tensor product $N \bar{\otimes} \langle TM, \mu_M \rangle$ as defined in (39) is isomorphic to the skew monoidal product $N * M$.*

Proof. The statement follows from the fact that

$$(42) \quad \begin{array}{ccccc} N * T^2 M & \xrightleftharpoons[N * \mu_M]{\mu_{N, TM}} & N * TM & \xrightarrow{\mu_{N,M}} & N * M \\ & \xleftarrow{N * T\eta_M} & & \xleftarrow{N * \eta_M} & \end{array}$$

is a split coequalizer in \mathcal{M} . □

Let $F : \mathcal{M} \rightarrow \mathcal{M}^T$ denote the free algebra functor, $FM = \langle TM, \mu_M \rangle$. Then it follows from the Lemma that there is a natural isomorphism

$$(43) \quad j_{N,M} : N \bar{\otimes} FM \xrightarrow{\sim} N * M \quad \text{such that} \quad j_{N,M} \circ \pi_{N, TM} = \mu_{N,M}$$

for each $N, M \in \mathcal{M}$. Using this isomorphism the coequalizer (39) becomes

$$(44) \quad N \bar{\otimes} FTM \xrightleftharpoons[N \bar{\otimes} F\alpha]{N \bar{\otimes} \mu_M} N \bar{\otimes} FM \xrightarrow{N \bar{\otimes} \alpha} N \bar{\otimes} \bar{M}$$

which shows that (39) is, up to isomorphism, the unique functor $\bar{\otimes} : \mathcal{M} \times \mathcal{M}^T \rightarrow \mathcal{M}$ which preserves reflexive coequalizers in the 2nd argument and for which $N \bar{\otimes} FM \cong N * M$.

By the way, the isomorphism $N * M \cong N \bar{\otimes} \langle TM, \mu_M \rangle$ is in complete agreement with the formula [19, Eqn. (30)] for the skew monoidal product of a bialgebroid. But for a perfect analogy we should be able to write $\langle TM, \mu_M \rangle$ as a ‘vertical tensor product’ $M \otimes H$ where $H = R * R$. The construction of such a vertical tensor product is still an open problem.

Although one would expect $\bar{\otimes}$ to be a monoidal product on the category \mathcal{M}^T of modules, this is not the case. It is only skew monoidal although its left unitor is always invertible as we will see soon. Whether $\bar{\otimes}$ can be indeed monoidal will be investigated in Section 8.

Theorem 4.2. *Let $\langle \mathcal{M}, *, R, \gamma, \eta, \varepsilon \rangle$ be a skew monoidal category with coequalizers of reflexive pairs and with $*$ preserving these coequalizers in the second argument. Then \mathcal{M}^T , the Eilenberg-Moore category of the canonical monad T , has reflexive coequalizers and there exist functors*

$$\mathcal{M} \times \mathcal{M}^T \rightarrow \mathcal{M}, \quad \mathcal{M}^T \times \mathcal{M}^T \rightarrow \mathcal{M}^T,$$

both of them denoted by $\bar{\otimes}$ and called the horizontal tensor product, such that

- (i) \mathcal{M}^T with $\bar{\otimes}$ as skew monoidal product is a skew monoidal category;
- (ii) \mathcal{M} with the action $\bar{\otimes} : \mathcal{M} \times \mathcal{M}^T \rightarrow \mathcal{M}$ is a right actegory over \mathcal{M}^T ;
- (iii) for each object $M \in \mathcal{M}$ the functor $M \bar{\otimes} -$ and for each T -algebra $\bar{M} \in \mathcal{M}^T$ the functor $\bar{M} \bar{\otimes} -$ preserves reflexive coequalizers;
- (iv) the left unitor $\bar{\eta}$ of the skew monoidal product $\bar{\otimes}$ on \mathcal{M}^T is invertible therefore the category of modules over \mathcal{M}^T is trivial.

Proof. The action $\bar{\otimes} : \mathcal{M} \times \mathcal{M}^T \rightarrow \mathcal{M}$ is defined by the coequalizer (39). Its associativity constraint can be obtained as follows. Let N be an object in \mathcal{M} and let $\bar{M} = \langle M, \alpha \rangle$ and $\bar{L} = \langle L, \beta \rangle$ be objects of \mathcal{M}^T . We need presentations for both the source and target of an arrow $N \bar{\otimes} (\bar{M} \bar{\otimes} \bar{L}) \rightarrow (N \bar{\otimes} \bar{M}) \bar{\otimes} \bar{L}$. Consider the following two 3x3 diagrams with all columns, and in the case of the first also the rows, being reflexive coequalizers:

$$(45) \quad \begin{array}{ccccc} N * T(M * TL) & \xrightarrow[N * T(M * \beta)]{N * T\mu_{M,L}} & N * T(M * L) & \xrightarrow{N * T\pi_{M,\bar{L}}} & N * T(M \bar{\otimes} \bar{L}) \\ \mu \Downarrow 1 * [(\alpha * 1) \circ \gamma] & & \mu \Downarrow 1 * [(\alpha * 1) \circ \gamma] & & \mu \Downarrow N * \psi \\ N * (M * TL) & \xrightarrow[N * (M * \beta)]{N * \mu_{M,L}} & N * (M * L) & \xrightarrow{N * \pi_{M,\bar{L}}} & N * (M \bar{\otimes} \bar{L}) \\ \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\ N \bar{\otimes} (\bar{M} \triangleleft TL) & \xrightarrow[N \bar{\otimes} (M * \beta)]{N \bar{\otimes} \mu_{M,L}} & N \bar{\otimes} (\bar{M} \triangleleft L) & \xrightarrow{N \bar{\otimes} \pi_{M,\bar{L}}} & N \bar{\otimes} (\bar{M} \bar{\otimes} \bar{L}) \end{array}$$

and

$$(46) \quad \begin{array}{ccccc} (N * TM) * TL & \xrightarrow[(N * \alpha) * TL]{\mu_{N,M} * TL} & (N * M) * TL & \xrightarrow{\pi_{N,\bar{M}} * TL} & (N \bar{\otimes} \bar{M}) * TL \\ \mu \Downarrow 1 * \beta & & \mu \Downarrow 1 * \beta & & \mu \Downarrow 1 * \beta \\ (N * TM) * L & \xrightarrow[(N * \alpha) * L]{\mu_{N,M} * L} & (N * M) * L & \xrightarrow{\pi_{N,\bar{M}} * L} & (N \bar{\otimes} \bar{M}) * L \\ \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\ (N * TM) \bar{\otimes} \bar{L} & \xrightarrow[(N * \alpha) \bar{\otimes} \bar{L}]{\mu_{N,M} \bar{\otimes} \bar{L}} & (N * M) \bar{\otimes} \bar{L} & \xrightarrow{\pi_{N,\bar{M}} \bar{\otimes} \bar{L}} & (N \bar{\otimes} \bar{M}) \bar{\otimes} \bar{L} \end{array}$$

Using the Diagonal Lemma [11, p.4] the diagonal of (45) is a coequalizer. This is the first row of the next diagram. The diagonal of (46) is just a "commuting" fork. It appears in the second row of

$$(47) \quad \begin{array}{ccccc} N * T(M * TL) & \xrightarrow[\quad 1 * [(\alpha * \beta) \circ \gamma] \quad]{\mu \circ (1 * T\mu)} & N * (M * L) & \xrightarrow{\pi \circ (1 * \pi)} & N \bar{\otimes} (\bar{M} \bar{\otimes} \bar{L}) \\ \downarrow \gamma \circ (1 * \gamma) & & \downarrow \gamma & & \downarrow \bar{\gamma} \\ (N * TM) * TL & \xrightarrow[\quad (1 * \alpha) * \beta \quad]{\mu \circ (\mu * 1)} & (N * M) * L & \xrightarrow{\pi \circ (\pi * 1)} & (N \bar{\otimes} \bar{M}) \bar{\otimes} \bar{L} \end{array}$$

with the unique arrow on the right hand side defining the associator $\bar{\gamma}_{N, \bar{M}, \bar{L}}$ we have been looking for. In order to facilitate the proof of the pentagon relation for $\bar{\gamma}$ let us observe that the left vertical arrow in (47) looks like an associator for a new skew monoidal product. This is almost the case.

Lemma 4.3. *If $\langle \mathcal{M}, *, R, \gamma, \eta, \varepsilon \rangle$ is a skew monoidal category then the data*

$$(48) \quad M *^2 N := M * TN$$

$$(49) \quad R^2 := R$$

$$(50) \quad \gamma_{L, M, N}^2 := \gamma_{L, TM, TN} \circ (L * \gamma_{R, M, TN})$$

$$(51) \quad \eta_M^2 := \eta_{TM} \circ \eta_M$$

$$(52) \quad \varepsilon_M^2 := \varepsilon_M \circ (M * \varepsilon_R)$$

obey 4 of the 5 skew monoidal category axioms: The failing axiom is the $\varepsilon\gamma\eta$ -triangle (27). But if η is epi then $\langle \mathcal{M}, *^2, R^2, \gamma^2, \eta^2, \varepsilon^2 \rangle$ is a skew monoidal category.

Continuing with the proof of the Theorem we need only one consequence of the Lemma, namely that γ^2 satisfies the pentagon equation. Since $\bar{\otimes}$ preserves reflexive coequalizers it is now easy to show that $\bar{\gamma}$, too, satisfies the pentagon equation

$$(53) \quad (\bar{\gamma}_{N, \bar{M}, \bar{L}} \bar{\otimes} \bar{K}) \circ \bar{\gamma}_{N, \bar{M} \bar{\otimes} \bar{L}, \bar{K}} \circ (N \bar{\otimes} \bar{\gamma}_{\bar{M}, \bar{L}, \bar{K}}) = \bar{\gamma}_{N \bar{\otimes} \bar{M}, \bar{L}, \bar{K}} \circ \bar{\gamma}_{N, \bar{M}, \bar{L} \bar{\otimes} \bar{K}}$$

where $\bar{\gamma}_{\bar{M}, \bar{L}, \bar{K}} : \bar{M} \bar{\otimes} (\bar{L} \bar{\otimes} \bar{K}) \rightarrow (\bar{M} \bar{\otimes} \bar{L}) \bar{\otimes} \bar{K}$ is the lifting of $\gamma_{M, L, K}$ to \mathcal{M}^T . Such a lifting exists since not only $\mu_{N, M}$, $N * \alpha$ and $\pi_{N, \bar{M}}$ can be lifted by (41) but also γ and γ^2 by (8). This lifted $\bar{\gamma}$ is nothing but the associator for $\bar{\otimes}$ as a skew monoidal product.

The next task is to study the unitors and their coherence conditions. As for the unit object for $\bar{\otimes}$ we define it to be the T -algebra $\bar{R} = \langle R, \varepsilon_R \rangle$.

The right unitor $\bar{\varepsilon}_N : N \bar{\otimes} \bar{R} \rightarrow N$ can be obtained from unique factorization of ε_N through the coequalizer $\pi_{N, \bar{R}}$

$$(54) \quad \bar{\varepsilon}_N \circ \pi_{N, \bar{R}} = \varepsilon_N, \quad N \in \text{ob } \mathcal{M}$$

since $\varepsilon_N \circ \mu_{N, R} = \varepsilon_N \circ (N * \varepsilon_R)$ is an identity in any skew monoidal category. For $\bar{N} = \langle N, \alpha \rangle$ we can define $\bar{\varepsilon}_{\bar{N}} : \bar{N} \bar{\otimes} \bar{R} \rightarrow \bar{N}$ simply as the lift to \mathcal{M}^T of $\bar{\varepsilon}_N$ since in the diagram

$$\begin{array}{ccccc} R * (N * R) & \xrightarrow{T\pi} & R * (N \bar{\otimes} \bar{R}) & \xrightarrow{T\bar{\varepsilon}_N} & R * N \\ (\alpha * 1) \circ \gamma \downarrow & & \downarrow \psi & & \downarrow \alpha \\ N * R & \xrightarrow{\pi} & N \bar{\otimes} \bar{R} & \xrightarrow{\bar{\varepsilon}_N} & N \end{array}$$

the left square is commutative because of (40), the outer rectangle is commutative because $\varepsilon_N \circ (\alpha * R) \circ \gamma_{R,N,R} = \alpha \circ \varepsilon_{TN} \circ \gamma_{R,N,R} = \alpha \circ T\varepsilon_N$ is an identity and $R * \pi_{N,\bar{R}}$ is epimorphic; so commutativity of the right square follows.

In order to prove the triangle relation (26) for $\bar{\otimes}$ we substitute $\bar{L} = \bar{R}$ in (47) and compose the second row with $\bar{\varepsilon}$. The calculation

$$\begin{aligned} & \bar{\varepsilon}_N \bar{\otimes} \bar{M} \circ \bar{\gamma}_{N,\bar{M},\bar{R}} \circ \pi_{N,\bar{M} \bar{\otimes} \bar{R}} \circ (N * \pi_{M,\bar{R}}) = \\ & \bar{\varepsilon}_N \bar{\otimes} \bar{M} \circ \pi_{N,\bar{M},\bar{R}} \circ (\pi_{N,\bar{M}} * R) \circ \gamma_{N,M,R} = \\ & \varepsilon_N \bar{\otimes} \bar{M} \circ (\pi_{N,\bar{M}} * R) \circ \gamma_{N,M,R} = \\ & \pi_{N,\bar{M}} \circ \varepsilon_{N*M} \circ \gamma_{N,M,R} = \\ & \pi_{N,\bar{M}} \circ (N * \varepsilon_M) = \\ & \pi_{N,\bar{M}} \circ (N * \bar{\varepsilon}_M) \circ (N * \pi_{M,\bar{R}}) = \\ & (N \bar{\otimes} \bar{\varepsilon}_M) \circ \pi_{N,\bar{M} \bar{\otimes} \bar{R}} \circ (N * \pi_{M,\bar{R}}) \end{aligned}$$

proves the coherence condition

$$(55) \quad \bar{\varepsilon}_N \bar{\otimes} \bar{M} \circ \bar{\gamma}_{N,\bar{M},\bar{R}} = N \bar{\otimes} \bar{\varepsilon}_M.$$

Lifting to \mathcal{M}^T we obtain for all T -algebras \bar{N} and \bar{M} that

$$(56) \quad \bar{\varepsilon}_{\bar{N}} \bar{\otimes} \bar{M} \circ \bar{\gamma}_{\bar{N},\bar{M},\bar{R}} = \bar{N} \bar{\otimes} \bar{\varepsilon}_M.$$

The left unitor $\bar{\eta}_{\bar{M}} : \bar{M} \rightarrow \bar{R} \bar{\otimes} \bar{M}$ is defined simply by the equation

$$(57) \quad \eta_{\bar{M}} := \pi_{R,\bar{M}} \circ \eta_M$$

which happens to be a T -algebra morphism in spite of that η_M is not. The outer rectangle of

$$\begin{array}{ccccc} TM & \xrightarrow{T\eta_M} & T^2M & \xrightarrow{T\pi_{R,\bar{M}}} & T(R \bar{\otimes} \bar{M}) \\ \alpha \downarrow & \bullet & \downarrow \mu_M & & \downarrow \psi \\ M & \xrightarrow{\eta_M} & TM & \xrightarrow{\pi_{R,\bar{M}}} & R \bar{\otimes} \bar{M} \end{array}$$

commutes although the left square does not.

The proof of the $\bar{\eta}$ -triangle

$$(58) \quad \bar{\gamma}_{\bar{R},\bar{M},\bar{N}} \circ \bar{\eta}_{\bar{M}} \bar{\otimes} \bar{N} = \bar{\eta}_{\bar{M}} \bar{\otimes} \bar{N}$$

goes as follows:

$$\begin{aligned} & \bar{\gamma}_{\bar{R},\bar{M},\bar{N}} \circ \bar{\eta}_{\bar{M}} \bar{\otimes} \bar{N} \circ \pi_{M,\bar{N}} = \\ & \bar{\gamma}_{\bar{R},\bar{M},\bar{N}} \circ \pi_{R,\bar{M} \bar{\otimes} \bar{N}} \circ \eta_{M \bar{\otimes} \bar{N}} \circ \pi_{M,\bar{N}} = \\ & \bar{\gamma}_{\bar{R},\bar{M},\bar{N}} \circ \pi_{R,\bar{M} \bar{\otimes} \bar{N}} \circ (R * \pi_{M,\bar{N}}) \circ \eta_{M * N} \stackrel{(47)}{=} \\ & \pi_{R \bar{\otimes} \bar{M},\bar{N}} \circ (\pi_{R,\bar{N}} * N) \circ \gamma_{R,M,N} \circ \eta_{M * N} = \\ & \pi_{R \bar{\otimes} \bar{M},\bar{N}} \circ (\pi_{R,\bar{N}} * N) \circ (\eta_M * N) = \\ & \pi_{R \bar{\otimes} \bar{M},\bar{N}} \circ (\bar{\eta}_M * N) = \\ & (\bar{\eta}_M \bar{\otimes} \bar{N}) \circ \pi_{M,\bar{N}}. \end{aligned}$$

The mixed triangle for $\bar{\otimes}$, i.e., the coherence condition

$$(59) \quad (\bar{\varepsilon}_N \bar{\otimes} \bar{M}) \circ \bar{\gamma}_{N,\bar{R},\bar{M}} \circ (N \bar{\otimes} \bar{\eta}_{\bar{M}}) = N \bar{\otimes} \bar{M}$$

and its lifted version

$$(60) \quad (\bar{\varepsilon}_{\bar{N}} \bar{\otimes} \bar{M}) \circ \bar{\gamma}_{\bar{N},\bar{R},\bar{M}} \circ (\bar{N} \bar{\otimes} \bar{\eta}_{\bar{M}}) = \bar{N} \bar{\otimes} \bar{M}$$

can be obtained as follows:

$$\begin{aligned}
& (\bar{\varepsilon}_N \bar{\otimes} \bar{M}) \circ \bar{\gamma}_{N, \bar{R}, \bar{M}} \circ (N \bar{\otimes} \bar{\eta}_{\bar{M}}) \circ \pi_{N, \bar{M}} = \\
& (\bar{\varepsilon}_N \bar{\otimes} \bar{M}) \circ \bar{\gamma}_{N, \bar{R}, \bar{M}} \circ \pi_{N, \bar{R} \bar{\otimes} \bar{M}} \circ (N * \pi_{R, \bar{M}}) \circ (N * \eta_M) = \\
& (\bar{\varepsilon}_N \bar{\otimes} \bar{M}) \circ \underbrace{\pi_{N \bar{\otimes} \bar{R}, \bar{M}} \circ (\pi_{N, \bar{R}} * M)}_{(\pi_{N, \bar{R}} \bar{\otimes} \bar{M}) \circ \pi_{N * R, \bar{M}}} \circ \gamma_{N, R, M} \circ (N * \eta_M) = \\
& (\varepsilon_N \bar{\otimes} \bar{M}) \circ \pi_{N * R, \bar{M}} \circ \gamma_{N, R, M} \circ (N * \eta_M) = \\
& \pi_{N, \bar{M}} \circ (\varepsilon_N * M) \circ \gamma_{N, R, M} \circ (N * \eta_M) = \\
& \pi_{N, \bar{M}}.
\end{aligned}$$

Finally, the axiom (5) for $\bar{\otimes}$, i.e., the relation

$$(61) \quad \bar{\varepsilon}_R \circ \bar{\eta}_R = \bar{R}$$

follows immediately from (57), (54) and (28). In this way the data $\langle \mathcal{M}^T, \bar{\otimes}, \bar{R}, \bar{\gamma}, \bar{\eta}, \bar{\varepsilon} \rangle$ is a skew monoidal category by the lifted version of (53) and by (58), (56), (60) and (61). This proves (i). The three axioms of Definition 2.2 for \mathcal{M} to be a right \mathcal{M}^T -actegory are provided by (53), (55) and (59). This proves (ii).

It should be clear from (39) and (41), that $\bar{\otimes}$, both as a tensor product and as an action, preserves all type of colimits that are preserved by $*$. Under the present assumptions this means that $\bar{\otimes}$ preserves at least the reflexive coequalizers in the 2nd argument. This proves (iii).

For any T -algebra $\bar{M} = \langle M, \alpha \rangle$ the canonical presentation

$$T^2 M \xrightleftharpoons[T\alpha]{\mu_M} TM \xrightarrow{\alpha} M$$

and the definition of the tensor $R \bar{\otimes} \bar{M}$

$$R * TM \xrightleftharpoons[R * \alpha]{\mu_{R, M}} R * M \xrightarrow{\pi_{R, \bar{M}}} R \bar{\otimes} \bar{M}$$

are coequalizers of the same reflexive pair. Therefore there exists an isomorphism i such that $i \circ \pi_{R, \bar{M}} = \alpha$. By the definition of $\bar{\eta}$ in (57)

$$i \circ \bar{\eta}_{\bar{M}} = i \circ \pi_{R, \bar{M}} \circ \eta_M = \alpha \circ \eta_M = M$$

hence $\bar{\eta}_{\bar{M}}$ is an isomorphism. Now an algebra for the monad $\bar{R} \bar{\otimes} -$ with underlying object, let us say, \bar{M} has unique action $\bar{\eta}_{\bar{M}}^{-1} : \bar{R} \bar{\otimes} \bar{M} \rightarrow \bar{M}$. This means that the category of modules over \mathcal{M}^T is \mathcal{M}^T itself. This proves (iv). \square

Proposition 4.4. *Under the assumptions of Theorem 4.2 the forgetful functor $G : \mathcal{M}^T \rightarrow \mathcal{M}$, $\langle M, \alpha \rangle \rightarrow M$, is a strict morphism of right \mathcal{M}^T -actegories and has the structure of a strictly normal skew monoidal functor.*

Proof. The way we lifted $N \bar{\otimes} \bar{M}$ to \mathcal{M}^T in (40) ensures that

$$G(\bar{N} \bar{\otimes} \bar{M}) = G\bar{N} \bar{\otimes} \bar{M}$$

which is precisely the statement that the strength of G as a right \mathcal{M}^T -actegory morphism is the identity. But more interesting is that the coequalizer in (39) can be read as a multiplicativity constraint for G ,

$$G_{\bar{N}, \bar{M}} := \pi_{G\bar{N}, \bar{M}} : G\bar{N} * G\bar{M} \rightarrow G(\bar{N} \bar{\otimes} \bar{M}).$$

The unit constraint is the identity $R = G\bar{R}$. The right square in (47) is just the hexagon expressing associativity of $G_{\bar{N}, \bar{M}}$ and the unitality tetragons reduce to triangles which are but the definitions (57) and (54) of $\bar{\eta}$ and $\bar{\varepsilon}$, respectively. \square

Combining the results of the above Theorem and Proposition with the \mathcal{V} -actegory structure we obtain the following.

Theorem 4.5. *Let \mathcal{M} be a $r2$ -exact skew monoidal \mathcal{V} -category. Then \mathcal{M}^T is also a $r2$ -exact skew monoidal \mathcal{V} -category and the forgetful functor $G : \mathcal{M}^T \rightarrow \mathcal{M}$ is a skew monoidal \mathcal{V} -functor which is strict as an actegory morphism.*

Proof. By Lemma 3.3, Theorem 4.2 and Proposition 4.4 we only have to show that

- (i) $\bar{\otimes}$ is a \mathcal{V} -functor, that is to say a 2-variable actegory morphism,
 - (ii) the partial functor $\bar{M} \bar{\otimes} -$ is strong;
 - (iii) $\bar{\gamma}$, $\bar{\eta}$ and $\bar{\varepsilon}$ are \mathcal{V} -natural, i.e., they are transformations of actegory morphisms.
- (i) The two strengths of $\bar{\otimes}$ can be introduced as the unique arrows making the diagrams

$$(62) \quad \begin{array}{ccccc} V \odot (N * TM) & \xrightleftharpoons[V \odot (N * \alpha)]{V \odot \mu_{N,M}} & V \odot (N * M) & \xrightarrow{V \odot \pi_{N,\bar{M}}} & V \odot (N \bar{\otimes} \bar{M}) \\ \downarrow \Gamma_{V,N,TM} & & \downarrow \Gamma_{V,N,M} & & \downarrow \bar{\Gamma}_{V,N,\bar{M}} \\ (V \odot N) * TM & \xrightleftharpoons[(V \odot N) * \alpha]{\mu_{V \odot N, M}} & (V \odot N) * M & \xrightarrow{\pi_{V \odot N, \bar{M}}} & (V \odot N) \bar{\otimes} \bar{M} \end{array}$$

and

$$(63) \quad \begin{array}{ccccc} V \odot (N * TM) & \xrightleftharpoons[V \odot (N * \alpha)]{V \odot \mu_{N,M}} & V \odot (N * M) & \xrightarrow{V \odot \pi_{N,\bar{M}}} & V \odot (N \bar{\otimes} \bar{M}) \\ \downarrow (N * \Gamma'_{V,R,M}) \circ \Gamma'_{V,N,TM} & & \downarrow \Gamma'_{V,N,M} & & \downarrow \bar{\Gamma}'_{V,N,\bar{M}} \\ N * T(V \odot M) & \xrightleftharpoons[N * \hat{\alpha}]{\mu_{N, V \odot M}} & N * (V \odot M) & \xrightarrow{\pi_{N, V \odot \bar{M}}} & N \bar{\otimes} (V \odot \bar{M}) \end{array}$$

commutative where in the second we used the notation $\hat{\alpha} := (V \odot \alpha) \circ \Gamma'^{-1}_{V,R,M}$. Calling the first vertical arrows of these diagrams $\Gamma^2_{V,N,M}$ and $\Gamma'^2_{V,N,M}$, respectively, we find, as in Lemma 4.3, that Γ^2 and Γ'^2 obey the 5 coherence conditions making them strengths for the \mathcal{V} -functor $*^2$. It follows now from the exactness conditions that $\bar{\Gamma}$ and $\bar{\Gamma}'$ satisfy the coherence conditions making them strengths for $\bar{\otimes}$.

(ii) Since the first 2 vertical arrows in (63) are isomorphisms, so is the third.

(iii) \mathcal{V} -naturality of $\bar{\gamma}$ means proving equations (34), (35) and (36) with γ , Γ and $*$ replaced by $\bar{\gamma}$, $\bar{\Gamma}$ and $\bar{\otimes}$, respectively. After composing these equations with the epimorphism $(V \odot \pi_{L,\bar{M} \bar{\otimes} \bar{N}}) \circ (V \odot (L * \pi_{\bar{M},\bar{N}}))$, using the relations (47), (62) or (63) and then (34) or (35) or (36), respectively, the barred versions of (34), (35) or (36) can be proven. One can similarly verify the barred versions of (37) or (38) expressing \mathcal{V} -naturality of $\bar{\eta}$ and $\bar{\varepsilon}$. \square

5. A LIFTING THEOREM

Let \mathcal{M} and \mathcal{M}' be skew monoidal categories and let $K : \mathcal{M} \rightarrow \mathcal{M}'$ be a skew monoidal functor. We assume that both \mathcal{M} and \mathcal{M}' have reflexive coequalizers and that both skew monoidal products preserve reflexive coequalizers in the second argument. Thus both module categories \mathcal{M}^T and \mathcal{M}'^T

have a skew monoidal structure by Theorem 4.2 and have skew monoidal forgetful functors G and G' as in the diagram

$$(64) \quad \begin{array}{ccc} \mathcal{M}^T & \xrightarrow[\text{?}]{\bar{K}} & \mathcal{M}'^{T'} \\ \downarrow G & & \downarrow G' \\ \mathcal{M} & \xrightarrow{K} & \mathcal{M}' \end{array}$$

We are asking whether a skew monoidal lifting \bar{K} of K exists that makes the diagram (strictly) commutative in the 2-category of skew monoidal categories. This problem is not simply a lifting problem in the 2-category of (r2-exact) skew monoidal categories because the canonical monad is not skew monoidal. (In fact T is closer to be skew opmonoidal than skew monoidal but it is neither of them.) But we do not claim the existence of \bar{K} for general monads, either.

Theorem 5.1. *Let \mathcal{M} and \mathcal{M}' be as above. Then any skew monoidal functor $K : \mathcal{M} \rightarrow \mathcal{M}'$ has a skew monoidal lift \bar{K} to the categories of modules.*

If K is a skew monoidal \mathcal{V} -functor with $\mathcal{M}, \mathcal{M}'$ being r2-exact skew monoidal \mathcal{V} -categories then the lift \bar{K} is also a skew monoidal \mathcal{V} -functor.

Proof. Merely as ordinary functors the existence of \bar{K} follows from the universal property of the forgetful functor G' as it was formulated in [14, Theorem 8.1]. Indeed, KG is a skew monoidal functor the domain of which is normal skew monoidal. Hence it must factor uniquely through G' as skew monoidal functors. Nevertheless it is instructive to look at the form \bar{K} takes due to the special form of KG .

For arbitrary monads T, T' the possible lifts of K are known to be in bijection with natural transformations $\kappa : T'K \rightarrow KT$ such that the pair $\langle K, \kappa \rangle$ forms a monad morphism [18] $\langle \mathcal{M}, T \rangle \rightarrow \langle \mathcal{M}', T' \rangle$. Then the lift associated to κ is given by

$$\bar{K} : \langle M, \alpha \rangle \mapsto \langle KM, K\alpha \circ \kappa_M \rangle.$$

A candidate for a monad morphism is provided by the skew monoidal structure of K [19, Lemma 2.7], namely $\kappa_M = K_{R,M} \circ (K_0 * KM)$. We claim that this choice does the job.

The skew monoidal structure of \bar{K} can be constructed as follows. The diagram valid for all $\bar{M}_i = \langle M_i, \alpha_i \rangle \in \mathcal{M}^T$, $i = 1, 2$,

$$(65) \quad \begin{array}{ccccc} KM_1 * T'KM_2 & \xrightarrow[\text{\scriptsize } KM_1 * (K\alpha_2) \circ \kappa_{M_2}]{\text{\scriptsize } \mu'_{KM_1, KM_2}} & KM_1 * KM_2 & \xrightarrow{\pi'_{\bar{K}\bar{M}_1, \bar{K}\bar{M}_2}} & \bar{K}\bar{M}_1 \bar{\otimes} \bar{K}\bar{M}_2 \\ \downarrow \text{\scriptsize } K_{M_1, TM_2} \circ (KM_1 * \kappa_{M_2}) & & \downarrow \text{\scriptsize } K_{M_1, M_2} & & \downarrow \text{\scriptsize } \bar{K}_{M_1, M_2} \\ K(M_1 * TM_2) & \xrightarrow[\text{\scriptsize } K(M_1 * \alpha_2)]{K\mu_{M_1, M_2}} & K(M_1 * M_2) & \xrightarrow{K\pi_{M_1, M_2}} & \bar{K}(\bar{M}_1 \bar{\otimes} \bar{M}_2) \end{array}$$

defines the components of \bar{K}_2 . The unit constraint for \bar{K} is defined by

$$(66) \quad \bar{K}_0 := \left(\langle R', \varepsilon'_{R'} \rangle \xrightarrow{K_0} \bar{K} \langle R, \varepsilon_R \rangle \right).$$

We leave it to the reader to check that $\langle \bar{K}, \bar{K}_2, \bar{K}_0 \rangle$ is a skew monoidal functor and makes (64) commutative.

As far as the \mathcal{V} -structure is concerned, the assumption that K is a skew monoidal \mathcal{V} -functor includes the assumption that the multiplicativity constraint K_2 is \mathcal{V} -natural in the second argument,

$$(67) \quad K_{L,V \odot M} \circ (KL * K_{V,M}) \circ \Gamma'_{V,KL,KM} = K\Gamma'_{V,L,M} \circ K_{V,L * M} \circ (V \odot K_{L,M}).$$

This relation for $L = R$ suffices to define the strength of \bar{K} . As a matter of fact,

$$V \odot \bar{K}\bar{M} \xrightarrow{\bar{K}_{V,\bar{M}}} \bar{K}(V \odot \bar{M})$$

can be introduced by

$$(68) \quad G'\bar{K}_{V,\bar{M}} := K_{V,G\bar{M}}$$

since $K_{V,M}$ is a well-defined arrow

$$\langle V \odot KM, (V \odot K\alpha) \circ (V \odot \kappa_M) \circ \Gamma'^{-1}_{V,R',KM} \rangle \xrightarrow{K_{V,M}} \langle K(V \odot M), K(V \odot \alpha) \circ K\Gamma'^{-1}_{V,R,M} \circ \kappa_{V \odot M} \rangle$$

in $\mathcal{M}'^{T'}$ due to commutativity of the diagram

$$\begin{array}{ccccccc} T'(V \odot KM) & \xleftarrow[\sim]{\Gamma'} & V \odot T'KM & \xrightarrow{V \odot \kappa_M} & V \odot KTM & \xrightarrow{V \odot K\alpha} & V \odot KM \\ T'K_{V,M} \downarrow & & & & K_{V,TM} \downarrow & & \downarrow K_{V,M} \\ T'K(V \odot M) & \xrightarrow{\kappa_{V \odot M}} & KT(V \odot M) & \xleftarrow[\sim]{K\Gamma'} & K(V \odot TM) & \xrightarrow{K(V \odot \alpha)} & K(V \odot M) \end{array}$$

which follows from (67). The coherence conditions for \bar{K} to be a \mathcal{V} -functor, i.e.,

$$(69) \quad \bar{K}_{V \odot W, \bar{M}} \circ a_{V,W,\bar{K}\bar{M}} = \bar{K}a_{V,W,\bar{M}} \circ \bar{K}_{V,W \odot \bar{M}} \circ (V \odot \bar{K}_{W,\bar{M}})$$

$$(70) \quad \bar{K}_{I,\bar{M}} \circ l_{\bar{K}\bar{M}} = \bar{K}l_{\bar{M}}$$

reduce to the analogous relations for K if we apply G' and use (68) and strictness of the \mathcal{V} -functor G , i.e., the identities $Ga_{V,W,\bar{M}} = a_{V,W,G\bar{M}}$ and $Gl_{\bar{M}} = l_{G\bar{M}}$.

It remains to show that the skew monoidal structure of \bar{K} constructed in the first part consists of \mathcal{V} -natural transformations. For \bar{K}_0 there is nothing to prove but for $\bar{K}_{\bar{M},\bar{N}}$ we have to prove \mathcal{V} -naturality in the first and the second argument,

$$(71) \quad \bar{K}\bar{\Gamma}_{V,\bar{M},\bar{N}} \circ \bar{K}_{V,\bar{M} \otimes \bar{N}} \circ (V \odot \bar{K}_{\bar{M},\bar{N}}) = \bar{K}_{V \odot \bar{M},\bar{N}} \circ (\bar{K}_{V,\bar{M}} \otimes \bar{K}_{\bar{N}}) \circ \bar{\Gamma}_{V,\bar{K}\bar{M},\bar{K}\bar{N}}$$

$$(72) \quad \bar{K}\bar{\Gamma}'_{V,\bar{M},\bar{N}} \circ \bar{K}_{V,\bar{M} \otimes \bar{N}} \circ (V \odot \bar{K}_{\bar{M},\bar{N}}) = \bar{K}_{\bar{M},V \odot \bar{N}} \circ (\bar{K}_{\bar{M}} \otimes \bar{K}_{V,\bar{N}}) \circ \bar{\Gamma}'_{V,\bar{K}\bar{M},\bar{K}\bar{N}}.$$

Fortunately, these relations immediately reduce to the analogous relations for K , e.g. to relation (67), after applying the forgetful strict \mathcal{V} -functor G' . This finishes the construction of the lifting \bar{K} of K as a skew monoidal \mathcal{V} -functor. \square

As an application of the above Lifting Theorem we can show that a mere category equivalence proven in [19, Theorem 5.3] is actually a skew monoidal equivalence. Let E be the endomorphism monoid of the skew monoidal unit R , so E is a monoid in \mathcal{V} . In [19] we have shown that any skew monoidal structure $\langle \mathcal{M}, *, R \rangle$ on \mathcal{M} induces a skew monoidal structure $\langle {}_E\mathcal{M}, *_q, \underline{R} \rangle$ on the category of E -objects and there is a skew monoidal forgetful functor $\phi : {}_E\mathcal{M} \rightarrow \mathcal{M}$. Let T^q denote the canonical monad of $\langle {}_E\mathcal{M}, *_q, \underline{R} \rangle$.

Theorem 5.2. *For any $r2$ -exact skew monoidal tensored \mathcal{V} -category $\langle \mathcal{M}, *, R \rangle$ there is a strictly commutative diagram of skew monoidal \mathcal{V} -functors*

$$\begin{array}{ccc} ({}_E\mathcal{M})^{T_q} & \xrightarrow[\sim]{\bar{\phi}} & \mathcal{M}^T \\ G^q \downarrow & & \downarrow G \\ {}_E\mathcal{M} & \xrightarrow{\phi} & \mathcal{M} \end{array}$$

in which $\bar{\phi}$ is an equivalence of skew monoidal \mathcal{V} -categories, in particular a strong skew monoidal functor.

Since this result somewhat deviates from the main topic of the paper, its proof is relegated to the Appendix.

6. THE UNDERLYING CATEGORY

The aim of this section is the construction of the analogue of the bimodule category $R\text{-}\mathbf{Mod}\text{-}R$ underlying the module category of a bialgebroid over R . Since \mathcal{M} plays the role of the right R -module category $\mathbf{Mod}\text{-}R$ and $R\text{-}\mathbf{Mod}\text{-}R$ is the Eilenberg-Moore category of the monad $R \odot _$ on $\mathbf{Mod}\text{-}R$, the natural candidate is the category ${}_E\mathcal{M}$ of left E -objects in \mathcal{M} where $E := \mathcal{M}(R, R)$ is the endomorphism monoid of the object R . However the (skew) monoidal structure of ${}_E\mathcal{M}$ should be independent of the given $*$ -structure on \mathcal{M} , just like the monoidal category $R\text{-}\mathbf{Mod}\text{-}R$ exists independently of what kind of R -bialgebroids we want to consider. This implies e.g. that the skew monoidal product $*_q$ of the Appendix has nothing to do with the skew monoidal product \otimes we are looking for. A skew monoidal structure $\langle {}_E\mathcal{M}, \otimes, \underline{R} \rangle$ on the category of E -objects which is determined solely by the \mathcal{V} -category structure of \mathcal{M} and by the position of the object R in \mathcal{M} is called the underlying category, provided an appropriate skew monoidal forgetful functor $\langle \mathcal{M}^T, \otimes, \bar{R} \rangle \rightarrow \langle {}_E\mathcal{M}, \otimes, \underline{R} \rangle$ exists.

We start with the following observation.

Proposition 6.1. *For every object R in a tensored \mathcal{V} -category \mathcal{M} there is a skew monoidal structure on \mathcal{M} with unit object R . Using the functor $H := \mathcal{M}(R, _): \mathcal{M} \rightarrow \mathcal{V}$ the skew monoidal product is*

$$M \bullet N := HM \odot N$$

and the coherence morphisms are

$$(73) \quad \dot{\gamma}_{L,M,N} = (H_{HL,M} \odot N) \circ a_{HL, HM, N}$$

$$(74) \quad \dot{\eta}_M = (i_R \odot M) \circ l_M$$

$$(75) \quad \dot{\varepsilon}_M = \text{ev}_{R,M}.$$

Furthermore, the functor H , equipped with the natural transformation

$$H_{HM,N} : HM \odot HN \rightarrow H(M \bullet N)$$

and the arrow $I \xrightarrow{i_R} HR$, is a skew monoidal functor $\langle \mathcal{M}, \bullet, R \rangle \rightarrow \langle \mathcal{V}, \odot, I \rangle$.

If \mathcal{M} has and $V \odot _$ preserves reflexive coequalizers for all $V \in \mathcal{V}$ then $\langle \mathcal{M}, \bullet, R \rangle$ is $r2$ -exact.

Proof. The proof is a straightforward application of the coherence conditions

$$(76) \quad Ha_{V,W,M} \circ H_{V,W \odot M} \circ (V \odot H_{W,M} = H_{V \odot W, M} \circ a_{V, W, HM}$$

$$(77) \quad H_{I,M} \circ l_{HM} = Hl_M$$

for the strength $H_{V,M} := \chi_{V,R,M}$ introduced in (16) and in particular the relations

$$\begin{aligned} H \text{ ev}_{R,M} \circ H_{HM,R} &= c_{R,R,M} \\ H \text{ ev}_{R,M} \circ H_{HM,R} \circ (HM \odot i_R) &= r_{HM}. \end{aligned}$$

The details are left to the reader. As far as the r2-exactness is concerned, if $V \odot _$ preserves reflexive coequalizers for all $V \in \mathcal{V}$ then clearly does $M \bullet _$ the same for all $M \in \mathcal{M}$. Thus, for r2-exactness of $\langle \mathcal{M}, \bullet, R \rangle$ we need only that the dotted Γ' be an isomorphism. Since \bullet is obtained from the action, as a 2-variable strong morphism $_ \odot _ : \mathcal{V} \times \mathcal{M} \rightarrow \mathcal{M}$, by composing with H on the left leg, strongness on the right leg remains untouched. Indeed, the strengths of \bullet are

$$(78) \quad \dot{\Gamma}_{V,M,N} = (H_{V,M} \odot N) \circ a_{V,HM,N} : V \odot (M \bullet N) \rightarrow (V \odot M) \bullet N$$

$$(79) \quad \dot{\Gamma}'_{V,M,N} = a_{HM,V,N}^{-1} \circ (s_{V,HM} \odot N) \circ a_{V,HM,N} : V \odot (M \bullet N) \rightarrow M \bullet (V \odot N).$$

□

Now we are ready to apply Theorem 4.2 to $\langle \mathcal{M}, \bullet, R \rangle$ and to its canonical monad $\dot{T} = R \bullet _ = E \odot _$.

Corollary 6.2. *Let \mathcal{M} be a tensored \mathcal{V} -category with reflexive coequalizers and assume that $V \odot _$ preserves reflexive coequalizers for all $V \in \mathcal{V}$. Then the category $\mathcal{M}^{\dot{T}}$ of modules over the skew monoidal category $\langle \mathcal{M}, \bullet, R \rangle$ is the category ${}_E\mathcal{M}$ of E -objects equipped with ‘horizontal’ tensor product $\underline{M} \underline{\otimes} \underline{N}$ defined by the coequalizer*

$$(80) \quad \mathcal{M}(R, M) \odot (E \odot N) \xrightarrow[HM \odot \lambda]{(c_{R,R,M} \odot N) \circ a_{HM,E,N}} \mathcal{M}(R, M) \odot N \xrightarrow{\omega_{M,N}} M \underline{\otimes} N$$

and with skew unit object $\underline{R} = \langle R, \text{ev}_{R,R} \rangle$.

Proof. The canonical monad of the dot-structure is $\dot{T} = E \odot _$ and it has multiplication

$$\begin{aligned} \dot{\mu}_{L,M} &\equiv (\dot{\varepsilon}_L \bullet M) \circ \dot{\gamma}_{L,R,M} = \\ &= (H \text{ ev}_{R,L} \odot M) \circ (H_{HL,R} \odot M) \circ a_{HL,E,M} = \\ &\stackrel{(19)}{=} (c_{R,R,L} \odot M) \circ a_{HL,E,M} \end{aligned}$$

which shows that (80) is identical to the coequalizer (39) defining the horizontal tensor product for E -objects.

The skew monoidal structure of ${}_E\mathcal{M}$ obtained by applying Theorem 4.2 to $\langle \mathcal{M}, \bullet, R \rangle$ has skew monoidal unit $\underline{R} = \langle R, \text{ev}_{R,R} \rangle$ and its coherence morphisms $\underline{\gamma}$, $\underline{\eta}$ and $\underline{\varepsilon}$ are the unique arrows making the diagrams

$$(81) \quad \begin{array}{ccc} L \bullet (M \bullet N) & \xrightarrow{\omega_{\underline{L}, \underline{M} \underline{\otimes} \underline{N}} \circ (L \bullet \omega_{\underline{M}, \underline{N}})} & \underline{L} \underline{\otimes} (\underline{M} \underline{\otimes} \underline{N}) \\ \dot{\gamma}_{L,M,N} \downarrow & & \downarrow \underline{\gamma}_{\underline{L}, \underline{M}, \underline{N}} \\ (L \bullet M) \bullet N & \xrightarrow{\omega_{\underline{L} \underline{\otimes} \underline{M}, \underline{N}} \circ (\omega_{\underline{L}, \underline{N}} \bullet N)} & (\underline{L} \underline{\otimes} \underline{M}) \underline{\otimes} \underline{N} \end{array}$$

$$\begin{array}{ccc} M & \xlongequal{\quad} & M \\ \dot{\eta}_M \downarrow & & \downarrow \underline{\eta}_M \\ R \bullet M & \xrightarrow{\omega_{\underline{R}, \underline{M}}} & \underline{R} \underline{\otimes} \underline{M} \end{array} \quad \begin{array}{ccc} M \bullet R & \xrightarrow{\omega_{\underline{M}, \underline{R}}} & \underline{M} \underline{\otimes} \underline{R} \\ \dot{\varepsilon}_M \downarrow & & \downarrow \underline{\varepsilon}_M \\ M & \xlongequal{\quad} & M \end{array}$$

commutative.

□

The skew monoidal structure of ${}_E\mathcal{M}$ as presented by the above Corollary does not allow to say too much about whether the coherence morphisms are invertible, except for $\underline{\eta}$, which is always invertible by Theorem 4.2 (i). In the next Proposition we give an alternative description for which, however, we need some preparations.

Let $\langle {}_E\mathcal{V}_E, \odot, E \rangle$ be the monoidal category of E - E -bimodules in \mathcal{V} . The coherence isomorphisms of ${}_E\mathcal{V}_E$ are denoted by \underline{a} , \underline{l} and \underline{r} , all oriented according to our skew monoidal convention.

The category of right E -modules \mathcal{V}_E is a right ${}_E\mathcal{V}_E$ -category in the usual way, with the action defined by a choice of coequalizers

$$V \odot (E \odot W) \rightrightarrows V \odot W \longrightarrow \underline{V} \odot_E \underline{W}$$

of reflexive pairs in \mathcal{V} for each $\underline{V} \in \mathcal{V}_E$ and $\underline{W} \in {}_E\mathcal{V}_E$. The tensor product $\underline{U} \odot_E \underline{W}$ of two E - E -bimodules can be chosen by lifting this action to ${}_E\mathcal{V}_E$ along the forgetful functor $\phi : {}_E\mathcal{V}_E \rightarrow \mathcal{V}_E$ in such a way that ϕ to become a strict morphism of right ${}_E\mathcal{V}_E$ -categories.

By a similar choice of coequalizers in \mathcal{M} we can define a functor $\mathcal{V}_E \times {}_E\mathcal{M} \rightarrow \mathcal{M}$, also denoted by \odot ,

$$V \odot (E \odot M) \xrightarrow[V \odot \lambda]{(\rho \odot M) \circ a_{V,E,M}} V \odot M \xrightarrow{\tau_{V,M}} \underline{V} \odot_E \underline{M}$$

for each $\underline{V} \in \mathcal{V}_E$ and $\underline{M} \in {}_E\mathcal{M}$. By \mathcal{V} -naturality of the reflexive pair we can lift the result to ${}_E\mathcal{M}$ if \underline{V} is an E - E -bimodule. In this way \odot_E becomes an action making ${}_E\mathcal{M}$ to a left ${}_E\mathcal{V}_E$ -category.

The coherence isomorphisms are denoted by $\underline{a}_{\underline{U},\underline{V},\underline{M}}$ and $\underline{l}_{\underline{M}}$ and satisfy the usual category axioms. For $\underline{V} \in {}_E\mathcal{V}_E$ and $\underline{M} \in {}_E\mathcal{M}$ the coequalizers $\tau_{\underline{V},\underline{M}}$ become components of a strength

$$(82) \quad \dot{G}_{\underline{V},\underline{M}} := \tau_{\underline{V},\underline{M}} : \phi \underline{V} \odot \dot{G} \underline{M} \rightarrow \dot{G}(\underline{V} \odot_E \underline{M})$$

for $\dot{G} : {}_E\mathcal{M} \rightarrow \mathcal{M}$ relative to $\phi : {}_E\mathcal{V}_E \rightarrow \mathcal{V}$. That is to say, $\phi \dot{G}$ is a morphism from the ${}_E\mathcal{V}_E$ -category ${}_E\mathcal{M}$ to the \mathcal{V} -category \mathcal{M} in the sense of Definition 2.7. The corresponding coherence condition (13) takes the form

$$(83) \quad \dot{G}_{\underline{V} \odot_E \underline{W}, \underline{M}} \circ (\phi \underline{V}, \underline{W} \odot \dot{G} \underline{M}) \circ a_{\phi \underline{V}, \phi \underline{W}, \dot{G} \underline{M}} = \dot{G}_{\underline{a}_{\underline{V}, \underline{W}, \underline{M}}} \circ \dot{G}_{\underline{V}, \underline{W} \odot_E \underline{M}} \circ (\phi \underline{V} \odot \dot{G} \underline{W}, \underline{M})$$

which will be needed soon. Disregarding from the fact that ${}_E\mathcal{V}_E$ is not symmetric, ${}_E\mathcal{M}$ is in fact a tensored ${}_E\mathcal{V}_E$ -category with $\mathcal{M}(\underline{M}, \underline{N}) : E \odot E^{\text{op}} \rightarrow \mathcal{V}$ playing the role of the ${}_E\mathcal{V}_E$ -valued hom.

Let J denote the \mathcal{V} -functor

$$J := \mathcal{M}(\underline{R}, -) : \mathcal{M} \rightarrow \mathcal{V}_E$$

whose value on the object M is the right E -module HM with action $c_{R,R,M} : HM \odot E \rightarrow HM$. Its strength is the lifting of $H_{V,M}$ to \mathcal{V}_E ,

$$J_{V,M} = \left(V \odot JM \equiv \langle V \odot HM, (V \odot c_{R,R,M}) \circ a_{V, HM, E}^{-1} \rangle \xrightarrow{H_{V,M}} \langle H(V \odot M), c_{R,R,V \odot M} \rangle \equiv J(V \odot M) \right).$$

J has a left adjoint $J^* := - \odot_E \underline{R} : \mathcal{V}_E \rightarrow \mathcal{M}$. The notation $\underline{\text{ev}}$ and $\underline{\text{coev}}$ will be used for the counit and unit of $J^* \dashv J$.

By definition, the object R is called a *dense generator* for the \mathcal{V} -category \mathcal{M} if J is fully faithful [5, Vol. 1].

We introduce also the functor $\mathcal{J} : {}_E\mathcal{M} \rightarrow {}_E\mathcal{V}_E$ which associates to \underline{M} the composite \mathcal{V} -functor $E \xrightarrow{\underline{M}} \mathcal{M} \xrightarrow{J} \mathcal{V}_E$. So, it is legitimate to write $\mathcal{J}\underline{M} = J\underline{M}$. The purpose of using a new notation

is that we want to make \mathcal{J} a ${}_E\mathcal{V}_E$ -functor. This can be done by requiring strict commutativity of the square

$$\begin{array}{ccc} {}_E\mathcal{V}_E({}_E\mathcal{M}) & \xrightarrow{\phi\dot{G}} & \mathcal{V}\mathcal{M} \\ \mathcal{J}\downarrow & & \downarrow H \\ {}_E\mathcal{V}_E({}_E\mathcal{V}_E) & \xrightarrow{\phi\phi} & \mathcal{V}\mathcal{V} \end{array}$$

in the 2-category **ACT**. This means, beyond the equality $H\dot{G} = \phi\mathcal{J}$ of functors, that the strength of \mathcal{J} is determined by

$$(84) \quad \begin{array}{ccc} \phi\underline{V} \odot \overbrace{H\dot{G}\underline{M}}^{\phi\mathcal{J}} & \xrightarrow{\phi\underline{V}, \mathcal{J}\underline{M}} & \phi(\underline{V} \odot_E \mathcal{J}\underline{M}) \\ H_{\phi\underline{V}, \dot{G}\underline{M}} \downarrow & & \downarrow \phi\mathcal{J}_{\underline{V}, \underline{M}} \\ H(\phi\underline{V} \odot \dot{G}\underline{M}) & \xrightarrow{H\dot{G}_{\underline{V}, \underline{M}}} & \underbrace{\phi\mathcal{J}(\underline{V} \odot_E \underline{M})}_{H\dot{G}} \end{array}$$

We want to tie the definition of $\underline{\otimes}$ to the definition of \odot by setting the coequalizer ω of (80) to be

$$(85) \quad \omega_{L, \underline{M}} := \tau_{JL, \underline{M}} \quad L \in \mathcal{M}, \underline{M} \in {}_E\mathcal{M}.$$

This choice allows the following pleasant representation of the coherence morphisms of $\underline{\otimes}$.

Proposition 6.3. *The skew monoidal structure $\langle {}_E\mathcal{M}, \underline{\otimes}, \underline{R}, \underline{\gamma}, \underline{\eta}, \underline{\varepsilon} \rangle$ on the category of E -objects can be chosen in such a way that the skew monoidal product is*

$$\underline{L} \underline{\otimes} \underline{M} := \mathcal{J}\underline{L} \odot_E \underline{M}$$

and the coherence morphisms are

$$(86) \quad \underline{\gamma}_{\underline{L}, \underline{M}, \underline{N}} = (\mathcal{J}\underline{J}\underline{L}, \underline{M} \odot_E \underline{N}) \circ \underline{a}_{J\underline{L}, J\underline{M}, \underline{N}}$$

$$(87) \quad \underline{\eta}_{\underline{M}} = \underline{l}_{\underline{M}}$$

$$(88) \quad \underline{\varepsilon}_{\underline{M}} = \underline{\text{ev}}_{\underline{R}, \underline{M}}.$$

These expressions imply the following invertibility properties:

- (i) $\underline{\eta}$ is an isomorphism.
- (ii) $\underline{\varepsilon}$ is an isomorphism if R is a dense generator in \mathcal{M} as a \mathcal{V} -category.
- (iii) $\underline{\gamma}$ is an isomorphism if and only if $\mathcal{J}\underline{J}\underline{L}, \underline{M} \odot_E \underline{N}$ is an isomorphism for all E -objects $\underline{L}, \underline{M}$ and \underline{N} .

Proof. The choice (85) clearly entails the equality $\underline{L} \underline{\otimes} \underline{M} := \mathcal{J}\underline{L} \odot_E \underline{M}$.

If $\underline{M} = \langle M, \lambda \rangle$ then by the equality $\omega_{R, \underline{M}} = \tau_{E, \underline{M}}$ the unique arrow u such that $u \circ \omega_{R, \underline{M}} = \lambda$ is at the same time $\underline{\eta}_{\underline{M}}^{-1}$, just as in the proof of Theorem 4.2 (i), and $\underline{l}_{\underline{M}}^{-1}$, by definition of the coherence isomorphism \underline{l} of the ${}_E\mathcal{V}_E$ -actegory ${}_E\mathcal{M}$. This proves (87) from which (i) is obvious.

The counit $\underline{\text{ev}}_{\underline{R}, \underline{M}}$ of $\mathcal{J}^* \dashv \mathcal{J}$ is related to the counit of $- \odot R \dashv H$ by

$$\dot{G}\underline{\text{ev}}_{\underline{R}, \underline{M}} \circ \tau_{JM, \underline{R}} = \text{ev}_{R, M}.$$

By (85) and by the 2nd diagram of (81) the same equation defines $\underline{\varepsilon}_{\underline{M}}$. This proves (88) from which (ii) immediately follows.

In order to prove (86) we recall the proof of Corollary 6.2 that $\underline{\gamma}$ is defined by the equation

$$(89) \quad \dot{G}\underline{\gamma}_{\underline{L}, \underline{M}, \underline{N}} \circ \omega_{\underline{L}, \underline{M} \otimes \underline{N}} \circ (\dot{G}\underline{L} \bullet \omega_{\underline{M}, \underline{N}}) = \omega_{\underline{L} \otimes \underline{M}, \underline{N}} \circ (\omega_{\underline{L}, \underline{N}} \bullet \dot{G}\underline{N}) \circ \dot{\gamma}_{\dot{G}\underline{L}, \dot{G}\underline{M}, \dot{G}\underline{N}}.$$

It suffices to show that the expression (86) satisfies this equation. Substituting $\omega_{\underline{M}, \underline{N}} = \tau_{J\underline{M}, \underline{N}} = \dot{G}_{J\underline{M}, \underline{N}}$ and the expression (73) into the LHS of (89) the claim follows from the following computation:

$$\begin{aligned} & \dot{G}(\mathcal{J}_{J\underline{L}, \underline{M}} \odot_E \underline{N}) \circ \dot{G}a_{J\underline{L}, J\underline{M}, \underline{N}} \circ \dot{G}_{J\underline{L}, J\underline{M} \odot_E \underline{N}} \circ (\phi_{J\underline{L}} \odot \dot{G}_{J\underline{M}, \underline{N}}) \\ & \stackrel{(83)}{=} \dot{G}(\mathcal{J}_{J\underline{L}, \underline{M}} \odot_E \underline{N}) \circ \dot{G}_{J\underline{L} \odot_E J\underline{M}, \underline{N}} \circ (\phi_{J\underline{L}, J\underline{M}} \odot \dot{G}\underline{N}) \circ a_{\phi_{J\underline{L}}, \phi_{J\underline{M}}, \dot{G}\underline{N}} \\ & = \dot{G}_{J(J\underline{L} \odot_E \underline{M}), \underline{N}} \circ (\phi_{J\underline{L}, \underline{M}} \odot \dot{G}\underline{N}) \circ (\phi_{J\underline{L}, J\underline{M}} \odot \dot{G}\underline{N}) \circ a_{H\underline{G}\underline{L}, H\underline{G}\underline{M}, \dot{G}\underline{N}} \\ & \stackrel{(84)}{=} \dot{G}_{J(J\underline{L} \odot_E \underline{M}), \underline{N}} \circ (H\underline{G}_{J\underline{L}, \underline{M}} \odot \dot{G}\underline{N}) \circ (H_{H\underline{G}\underline{L}, \dot{G}\underline{M}} \odot \dot{G}\underline{N}) \circ a_{H\underline{G}\underline{L}, H\underline{G}\underline{M}, \dot{G}\underline{N}} \end{aligned}$$

which is the RHS of (89) by (73). This finishes the proof of (86) from which (iii) is obvious. \square

Remark 6.4. The coherence conditions for the strength $\mathcal{J}_{\underline{V}, \underline{M}} : \underline{V} \odot_E J\underline{M} \rightarrow J(\underline{V} \odot_E \underline{M})$ are

$$(90) \quad \mathcal{J}_{\underline{V} \odot_E \underline{W}, \underline{M}} \circ a_{\underline{V}, \underline{W}, J\underline{M}} = J a_{\underline{V}, \underline{W}, \underline{M}} \circ \mathcal{J}_{\underline{V}, \underline{W} \odot_E \underline{M}} \circ (\underline{V} \odot_E \mathcal{J}_{\underline{W}, \underline{M}})$$

$$(91) \quad \mathcal{J}_{\underline{E}, \underline{M}} \circ \underline{l}_{J\underline{M}} = J\underline{l}_{\underline{M}}$$

where $\underline{V}, \underline{W} \in {}_E\mathcal{V}_E$ and $\underline{M} \in {}_E\mathcal{M}$. The second relation immediately implies that $\mathcal{J}_{\underline{E}, \underline{M}}$ is an isomorphism and so is $\underline{\gamma}_{\underline{R}, \underline{M}, \underline{N}}$ for all $\underline{M}, \underline{N} \in {}_E\mathcal{M}$. This is in complete agreement with invertibility of $\underline{\eta}$ since the latter implies invertibility of $\underline{\gamma}_{\underline{R}, \underline{M}, \underline{N}}$ by the skew monoidal triangle axiom (25).

Similarly, if we assume invertibility of $\underline{\text{ev}}_{\underline{R}, -}$ the ${}_E\mathcal{V}_E$ -naturality relation (17) implies that $\mathcal{J}^* \mathcal{J}_{\underline{V}, \underline{M}}$ is invertible for all $\underline{V} \in {}_E\mathcal{V}_E$ and $\underline{M} \in {}_E\mathcal{M}$ and therefore $\underline{\gamma}_{\underline{L}, \underline{M}, \underline{R}}$ is an isomorphism for all $\underline{L}, \underline{M} \in {}_E\mathcal{M}$. This is again something that follows directly from invertibility of $\underline{\varepsilon}$ and the skew monoidal triangle (26).

Still assuming invertibility of $\underline{\text{ev}}_{\underline{R}, -}$ the $\underline{\text{coev}}_{\underline{R}, J\underline{M}}$ is an isomorphism by the adjunction relation. Hence the ${}_E\mathcal{V}_E$ -naturality condition (18) implies that

$$(92) \quad \underline{\text{coev}}_{\underline{R}, J\underline{L} \odot_E J\underline{M}} \text{ isomorphism} \Leftrightarrow \mathcal{J}_{J\underline{L}, \underline{M}} \text{ isomorphism}$$

for any pair of objects $\underline{L}, \underline{M} \in {}_E\mathcal{M}$.

The latter equivalence together with the observation that the skew monoidal structure of Proposition 6.3 is a special case of the Altenkirch-Chapman-Uustalu construction [3] will be the clue to characterize in Section 8 the situation of the underlying category being monoidal and $\mathcal{J} : {}_E\mathcal{M} \rightarrow {}_E\mathcal{V}_E$ being a strong monoidal functor.

7. THE FORGETFUL FUNCTOR

This section is about the forgetful functor that, for a (right) R -bialgebroid B , maps a right B -module to the underlying R - R -bimodule. This functor is strong monoidal and one of the main questions of this paper is whether similar forgetful functors exist on module categories \mathcal{M}^T of $r2$ -exact skew monoidal \mathcal{V} -categories \mathcal{M} .

The target of the forgetful functor has been constructed in the previous section. Therefore the functor \mathcal{G} we are looking for is of the type $\mathcal{M}^T \rightarrow {}_E\mathcal{M}$. Since both the target and source are Eilenberg-Moore categories of canonical monads, it seems natural to define \mathcal{G} by lifting a very elementary functor $\mathcal{M} \rightarrow \mathcal{M}$ which merely connects $*$ with \bullet .

Lemma 7.1. *Let $\langle \mathcal{M}, *, R \rangle$ be a skew monoidal tensored \mathcal{V} -category and let $\langle \mathcal{M}, \bullet, R \rangle$ be the skew monoidal \mathcal{V} -category structure associated to the tensored \mathcal{V} -category \mathcal{M} and to the object R by Proposition 6.1. Then the identity functor $\mathcal{M} \rightarrow \mathcal{M}$ together with the \mathcal{V} -natural transformation*

$$\sigma_{M,N} := \left(M \bullet N \xrightarrow{HM \odot \eta_N} HM \odot TN \xrightarrow{\Gamma_{HM,R,N}} (HM \odot R) * N \xrightarrow{\text{ev}_{R,M} * N} M * N \right)$$

and the identity arrow $R \rightarrow R$ is a skew monoidal \mathcal{V} -functor $\langle \mathcal{M}, *, R \rangle \xrightarrow{\Sigma} \langle \mathcal{M}, \bullet, R \rangle$.

Proof. The skew monoidal functor axioms

$$(93) \quad \gamma_{L,M,N} \circ \sigma_{L,M*N} \circ (L \bullet \sigma_{M,N}) = (\sigma_{L,M} * N) \circ \sigma_{L \bullet M, N} \circ \dot{\gamma}_{L,M,N}$$

$$(94) \quad \sigma_{R,M} \circ \dot{\eta}_M = \eta_M$$

$$(95) \quad \varepsilon_M \circ \sigma_{M,R} = \dot{\varepsilon}_M$$

can be verified by a straightforward calculation. \square

By [19, Lemma 2.7] the above Lemma implies that

$$\sigma_M := \sigma_{R,M} : R \bullet M \rightarrow R * M$$

is a monad morphism from $E \odot _$ to T . Therefore we define the forgetful functor \mathcal{G} as the functor induced by σ_M on the Eilenberg-Moore categories,

$$(96) \quad \mathcal{G} : \mathcal{M}^T \rightarrow {}_E\mathcal{M}, \quad \langle M, \alpha \rangle \mapsto \langle M, \alpha \circ \sigma_M \rangle.$$

In other words, \mathcal{G} is the lifting, in the sense of Theorem 5.1, of the functor defined in Lemma 7.1.

Theorem 7.2. *Let \mathcal{M} be an $r2$ -exact skew monoidal tensored \mathcal{V} -category. Then the forgetful functor \mathcal{G} defined in (96) is a skew monoidal \mathcal{V} -functor making the diagram*

$$\begin{array}{ccc} \mathcal{M}^T & \xrightarrow{\mathcal{G}} & {}_E\mathcal{M} \\ G \downarrow & & \downarrow \dot{G} \\ \mathcal{M} & \xrightarrow{\Sigma} & \mathcal{M} \end{array}$$

strictly commutative as skew monoidal \mathcal{V} -functors. Furthermore, \mathcal{G} is monadic as a \mathcal{V} -functor giving rise to a \mathcal{V} -monad (neither skew monoidal nor skew opmonoidal in general) \underline{T} on ${}_E\mathcal{M}$ such that $\mathcal{M}^T \cong ({}_E\mathcal{M})^{\underline{T}}$.

Proof. \mathcal{G} is the lift in the sense of Theorem 5.1 of the functor $\Sigma = \langle \mathcal{M}, \sigma, R \rangle$ constructed in Lemma 7.1. Therefore it is a skew monoidal \mathcal{V} -functor such that $\dot{G}\mathcal{G} = \Sigma G$. Merely as functors, however, we have $\dot{G}\mathcal{G} = G$ with both G and \dot{G} monadic. Since all categories have reflexive coequalizers, it follows by standard arguments relying on Beck's Theorem that \mathcal{G} is monadic precisely if it has a left adjoint.

As for the existence of the left adjoint \mathcal{F} we can rely on classical adjoint lifting theorems such as [4, Theorem 3.7.3]. It is interesting that a left adjoint can be explicitly given in terms of the quotient skew monoidal product (127) as

$$(97) \quad \mathcal{F} : {}_E\mathcal{M} \rightarrow \mathcal{M}^T, \quad \mathcal{F}\underline{M} = \langle R *_q \underline{M}, \nabla_{\mathcal{F}\underline{M}} \rangle$$

where the T -algebra structure is given by unique factorization in the diagram

$$(98) \quad \begin{array}{ccccc} E \odot (R * (R * M)) & \xrightleftharpoons[\lambda_3]{\rho_2} & R * (R * M) & \xrightarrow{Tq_{R,\underline{M}}} & T(R *_q \underline{M}) \\ \downarrow E \odot \mu_{R,M} & & \downarrow \mu_{R,M} & & \downarrow \nabla_{\mathcal{F}\underline{M}} \\ E \odot (R * M) & \xrightleftharpoons[\lambda_2]{\rho} & R * M & \xrightarrow{q_{R,\underline{M}}} & R *_q \underline{M} \end{array}$$

where the notation ρ_i, λ_i is that of Appendix A. Notice that the first row is not exactly the T of the second, it is modified by a harmless composition with Γ' (which is an isomorphism due to one of the r2-exactness conditions).

In order to make \mathcal{F} a \mathcal{V} -functor it suffices to supply it with a strength $\mathcal{F}_{V,\underline{M}} : V \odot \mathcal{F}\underline{M} \rightarrow \mathcal{F}(V \odot \underline{M})$. This can be done in the more general case of diagram (129), so $\mathcal{F}_{V,\underline{M}} = \Gamma'_{V,R,\underline{M}}^q$. This strength is in fact invertible, because Γ' is, in complete agreement with the fact that a left adjoint category morphism must always be strong. The ordinary adjunction $\mathcal{F} \dashv \mathcal{G}$ follows from [4, Theorem 3.7.3] with the remark that the existence of reflexive coequalizers suffices. \square

In the remaining part of this section we study the question of when \mathcal{G} is strong skew monoidal. Since $\mathcal{G}_0 : \underline{R} \rightarrow \mathcal{G}\underline{R}$ is the identity by (66) and by Σ_0 being the identity, the question reduces to studying $\mathcal{G}_{\bar{M},\bar{N}}$ the definition of which, according to (65), can be read from the diagram

$$(99) \quad \begin{array}{ccccc} M \bullet (R \bullet N) & \xrightleftharpoons[M \bullet \lambda_{\mathcal{G}\bar{N}}]{\dot{\mu}_{M,N}} & M \bullet N & \xrightarrow{\omega_{M,\mathcal{G}\bar{N}}} & M \underline{\otimes} \mathcal{G}\bar{N} \\ \downarrow \sigma_{M,TN} \circ (M \bullet \sigma_N) & & \downarrow \sigma_{M,N} & & \downarrow \mathcal{G}_{M,\bar{N}} \\ M * (R * N) & \xrightleftharpoons[M * \nabla_{\bar{N}}]{\mu_{M,N}} & M * N & \xrightarrow{\pi_{M,\bar{N}}} & M \bar{\otimes} \bar{N} \end{array}$$

by lifting $\mathcal{G}_{M,\bar{N}} \in \mathcal{M}$ to $\mathcal{G}_{\bar{M},\bar{N}} \in {}_E\mathcal{M}$. The unlifted $\mathcal{G}_{M,\bar{N}}$ -s are the components of a strength for the identity functor $\text{id}_{\mathcal{M}}$ that forms, together with the skew monoidal \mathcal{G} , an category morphism from the right \mathcal{M}^T -category \mathcal{M} to the right ${}_E\mathcal{M}$ -category \mathcal{M} (a right category version of Definition 2.7). Let us introduce the notation

$$\lambda_{M,N} := (\text{ev}_{R,M} * N) \circ \Gamma_{HM,R,N} : M \bullet (R * N) \rightarrow M * N$$

so that $\sigma_{M,N} = \lambda_{M,N} \circ (M \bullet \eta_N)$. The special case $\lambda_{R,N} : E \odot TN \rightarrow TN$ returns the E -object obtained from the free T -algebra $FN = \langle TN, \mu_N \rangle$ by applying \mathcal{G} . Indeed, $\mu_N \circ \sigma_{TN} = \lambda_{R,N}$. In

fact more is true:

$$\begin{aligned}
 \mu_{M,N} \circ \sigma_{M,TN} &= (\varepsilon_M * N) \circ \gamma_{M,R,N} \circ (\text{ev}_{R,M} * TN) \circ \Gamma_{HM,R,TN} \circ (HM \odot \eta_{TN}) = \\
 &= (\varepsilon_M * N) \circ ((\text{ev}_{R,M} * R) * N) \circ \gamma_{HM \odot R,R,N} \circ \Gamma_{HM,R,R*N} \circ (HM \odot \eta_{TN}) = \\
 &\stackrel{(34)}{=} (\text{ev}_{R,M} * N) \circ (\varepsilon_{HM \odot R} * N) \circ (\Gamma_{HM,R,R} * N) \circ \Gamma_{HM,R*R,N} \circ \\
 &\circ (HM \odot \gamma_{R,R,N}) \circ (HM \odot \eta_{TN}) = \\
 &\stackrel{(25)+(38)}{=} (\text{ev}_{R,M} * N) \circ ((HM \odot \varepsilon_R) * N) \circ \Gamma_{HM,R*R,N} \circ (HM \odot (\eta_R * N)) = \\
 &= (\text{ev}_{R,M} * N) \circ \Gamma_{HM,R,N} = \\
 (100) \quad &= \lambda_{M,N}
 \end{aligned}$$

for all objects M, N of \mathcal{M} .

Proposition 7.3. *For an r2-exact skew monoidal tensored \mathcal{V} -category \mathcal{M} the strength $\mathcal{G}_{M,\bar{N}} : M \underline{\otimes} \mathcal{G}\bar{N} \rightarrow M \bar{\otimes} \bar{N}$ is invertible if and only if $\lambda_{M,N}$ is a coequalizer of the reflexive pair*

$$M \bullet (R \bullet TN) \xrightleftharpoons[M \bullet \lambda_{R,N}]{\dot{\mu}_{M,TN}} M \bullet TN$$

for all $M, N \in \mathcal{M}$.

Proof. Inserting a free T -algebra $FN = \langle TN, \mu_N \rangle$ for \bar{N} in the definition (99) of $\mathcal{G}_{M,\bar{N}}$ and composing it with the isomorphism (43) we obtain by (100) that

$$j_{M,N} \circ \mathcal{G}_{M,FN} \circ \omega_{M,GFN} = j_{M,N} \circ \pi_{M,FN} \circ \sigma_{M,TN} = \mu_{M,N} \circ \sigma_{M,TN} = \lambda_{M,N}.$$

Therefore $\mathcal{G}_{M,FN}$ is invertible precisely if $\lambda_{M,N}$ is a coequalizer of the above pair. In this way we reduced the problem to prove invertibility of $\mathcal{G}_{M,\bar{N}}$ for all $\bar{N} \in \mathcal{M}^T$ whenever invertibility of $\mathcal{G}_{M,FN}$ is known for all $N \in \mathcal{M}$. Writing any T -algebra \bar{N} as a reflexive coequalizer of free T -algebras and using naturality of $\mathcal{G}_{M,\bar{N}}$ we have a commutative diagram

$$(101) \quad \begin{array}{ccccc}
 M \underline{\otimes} \mathcal{G}FTN & \xrightleftharpoons[M \underline{\otimes} \mathcal{G}T\nabla_{\bar{N}}]{M \underline{\otimes} \mathcal{G}\mu_N} & M \underline{\otimes} \mathcal{G}FN & \xrightarrow{M \underline{\otimes} \mathcal{G}\nabla_{\bar{N}}} & M \underline{\otimes} \mathcal{G}\bar{N} \\
 \downarrow \mathcal{G}_{M,FTN} & & \downarrow \mathcal{G}_{M,FN} & & \downarrow \mathcal{G}_{M,\bar{N}} \\
 M \bar{\otimes} FTN & \xrightleftharpoons[M \bar{\otimes} T\nabla_{\bar{N}}]{M \bar{\otimes} \mu_N} & M \bar{\otimes} FN & \xrightarrow{M \bar{\otimes} \nabla_{\bar{N}}} & M \bar{\otimes} \bar{N}
 \end{array}$$

The second row is a coequalizer since $\bar{\otimes}$ is r2-exact and this leads to a unique (dashed) arrow that makes the diagram, with $\mathcal{G}_{M,\bar{N}}$ removed, commutative. The first row is the result of applying $M \underline{\otimes} \mathcal{G}_-$ to a reflexive coequalizer which is in fact G -contractible. Using the relation $\dot{G}\mathcal{G} = G$ we see that \mathcal{G} maps G -contractible pairs to \dot{G} -contractible ones, so by monadicity of \dot{G} the arrow $\mathcal{G}\nabla_{\bar{N}}$ is the coequalizer of a reflexive pair in ${}_E\mathcal{M}$. By r2-exactness of $\underline{\otimes}$ we conclude that the first row is also a coequalizer, hence the dashed arrow is necessarily the inverse of $\mathcal{G}_{M,\bar{N}}$. \square

Corollary 7.4. *Let $\langle \mathcal{M}, *, R \rangle$ be an r-exact skew monoidal \mathcal{V} -category in which R is a dense generator for \mathcal{M} . Then $\mathcal{G} : \mathcal{M}^T \rightarrow {}_E\mathcal{M}$ is a strong skew monoidal functor.*

Proof. The statement follows by considering the commutative diagram

$$(102) \quad \begin{array}{ccccc} HL \odot (E \odot TM) & \xrightarrow[(HL \odot \lambda_{R,M})]{(c_{R,R,L} \odot TM) \circ a_{HL,E,TM}} & HL \odot TM & \xrightarrow{\lambda_{L,M}} & L * M \\ \downarrow \Gamma \circ (1 \odot \Gamma) & & \downarrow \Gamma & & \parallel \\ (HL \odot (E \odot R)) * M & \xrightarrow[(HL \odot \text{ev}_{R,R}) * M]{[(c_{R,R,L} \odot R) \circ a] * M} & (HL \odot R) * M & \xrightarrow{\text{ev}_{R,L} * M} & L * M \end{array}$$

in which the 2nd row is obtained by applying $- * M$ to a reflexive coequalizer. The vertical arrows are all isomorphisms by the assumption that $*$ is strong in the 1st argument. Thus the first row is a coequalizer, too, and Proposition 7.3 applies. \square

The following Theorem just summarizes the content of this Section.

Theorem 7.5. *Let $\langle \mathcal{M}, *, R \rangle$ be an r -exact skew monoidal tensored \mathcal{V} -category and assume that R is a dense generator for \mathcal{M} . Then the forgetful functor $\mathcal{G} : \mathcal{M}^T \rightarrow {}_E\mathcal{M}$ is strong skew monoidal and monadic and $\underline{T} = \mathcal{G}\mathcal{F}$ is a skew opmonoidal \mathcal{V} -monad on ${}_E\mathcal{M}$.*

8. SELF-COCOMPLETENESS AND MONOIDALITY

Assuming the forgetful functor $\mathcal{G} : \mathcal{M}^T \rightarrow {}_E\mathcal{M}$ is strong skew monoidal the question of whether $\bar{\otimes}$ on \mathcal{M}^T is a monoidal product can be answered affirmatively provided $\underline{\otimes}$ on ${}_E\mathcal{M}$ is monoidal. The latter being independent of the original skew monoidal product $*$ on \mathcal{M} we see that it is only the object R in the bare \mathcal{V} -category \mathcal{M} which we should study.

We consider a somewhat more general problem by replacing E with a small \mathcal{V} -category \mathcal{C} . In this way we will be able to comment also about the Altenkirch-Chapman-Uustalu skew monoidal structure on the functor category $[\mathcal{C}, \mathcal{M}]$. Also we relax, or just forget about, the existence of (arbitrary) tensors in \mathcal{M} . But we will need certain weighted colimits to exist in \mathcal{M} . In the absence of tensors the colimit $U \star F$ of $F : \mathcal{C} \rightarrow \mathcal{M}$ weighted by $U : \mathcal{C}^{\text{op}} \rightarrow \mathcal{V}$ cannot be represented by a coend $\int^{\mathcal{C}} UC \odot FC$. Instead it is defined [12] as the representing object in

$$\mathcal{M}(U \star F, M) \cong [\mathcal{C}^{\text{op}}, \mathcal{V}](U, \mathcal{M}(F -, M)), \quad M \in \mathcal{M}.$$

Definition 8.1. Let \mathcal{C} be a small \mathcal{V} -category. A full replete subcategory $\mathcal{W} \subseteq [\mathcal{C}^{\text{op}}, \mathcal{V}]$ is called self-cocomplete if

- (sc-1) \mathcal{W} contains all the representable functors $\mathcal{C}(-, C)$, $C \in \mathcal{C}$ and
- (sc-2) \mathcal{W} is closed under \mathcal{W} -weighted colimits (of \mathcal{V} -functors $\mathcal{C} \rightarrow \mathcal{W}$).

Remark 8.2. Since \mathcal{W} is full, condition (sc-1) is equivalent to that the Yoneda embedding $C \mapsto YC = \mathcal{C}(-, C)$ factors through the inclusion $\mathcal{W} \subseteq [\mathcal{C}^{\text{op}}, \mathcal{V}]$ via a unique (fully faithful) $Y_{\mathcal{W}} : \mathcal{C} \rightarrow \mathcal{W}$. Condition (sc-2), in turn, says that $U \star F$, which always exists in $[\mathcal{C}^{\text{op}}, \mathcal{V}]$ and can be represented there as a pointwise coend $D \mapsto \int^{\mathcal{C}} UC \odot FCD$, belongs to \mathcal{W} for all $U \in \mathcal{W}$ and $F \in [\mathcal{C}, \mathcal{W}]$. Using the fact that fully faithful functors reflect colimits the weighted colimit $U \star F$ is not only a colimit in $[\mathcal{C}^{\text{op}}, \mathcal{V}]$ but also in \mathcal{W} , provided $U \in \mathcal{W}$, $F \in [\mathcal{C}, \mathcal{W}]$. Thus (sc-2) can also be read as that \mathcal{W} has \mathcal{W} -weighted colimits and the inclusion $\mathcal{W} \subseteq [\mathcal{C}^{\text{op}}, \mathcal{V}]$ preserves them. Even if colimits, other than \mathcal{W} -weighted ones, happen to exist in \mathcal{W} these need not be preserved by the inclusion.

Lemma 8.3. *For $\mathcal{W} \subseteq [\mathcal{C}^{\text{op}}, \mathcal{V}]$ a self-cocomplete full replete subcategory and for a \mathcal{V} -functor $F : \mathcal{C} \rightarrow [\mathcal{C}^{\text{op}}, \mathcal{V}]$ the following conditions are equivalent:*

- (i) $U \star F \in \mathcal{W}$ for all $U \in \mathcal{W}$.

(ii) $FC \in \mathcal{W}$ for all $C \in \mathcal{C}$.

Proof. Since \mathcal{W} is full, (ii) implies that F factors through the inclusion $\mathcal{W} \subseteq [\mathcal{C}^{\text{op}}, \mathcal{V}]$ therefore (i) follows from assumption (sc-2) on \mathcal{W} . Vice versa, if (i) holds then setting U to be representable $FC \cong YC \star F \in \mathcal{W}$ for all C . \square

Corollary 8.4. For $\mathcal{W} \subseteq [\mathcal{C}^{\text{op}}, \mathcal{V}]$ a full, replete and self-cocomplete subcategory define $\underline{\mathcal{W}} := [\mathcal{C}, \mathcal{W}]$. Then $\underline{\mathcal{W}}$ is a monoidal subcategory of the bimodule category $[\mathcal{C} \odot \mathcal{C}^{\text{op}}, \mathcal{V}]$ via the embedding

$$[\mathcal{C}, \mathcal{W}] \subseteq [\mathcal{C}, [\mathcal{C}^{\text{op}}, \mathcal{V}]] \cong [\mathcal{C} \odot \mathcal{C}^{\text{op}}, \mathcal{V}]$$

and \mathcal{W} is a right $\underline{\mathcal{W}}$ -actegory via the weighted colimit operation \star .

Proof. $\underline{\mathcal{W}}$ consists precisely of the functors $F : \mathcal{C} \mapsto \{D \mapsto F(C, D) \in \mathcal{V}\}$ satisfying condition (ii) of Lemma 8.3. By condition (i) of the same Lemma it is clear that $\underline{\mathcal{W}}$ is monoidal with monoidal product

$$(F \otimes G)(B, D) = \int^C F(B, C) \odot G(C, D)$$

and with monoidal unit $Y_{\mathcal{W}}(C, D) = \mathcal{C}(D, C)$. \square

Example 8.5. For \mathcal{V} -functors $U : \mathcal{C}^{\text{op}} \rightarrow \mathcal{V}$ and $X : \mathcal{D} \rightarrow \mathcal{V}$ with \mathcal{C} and \mathcal{D} small the colimits $U \star _$ weighted by U exist both in \mathcal{V} and $[\mathcal{D}, \mathcal{V}]$ and the limits $\{X, _ \}$ weighted by X exist both in \mathcal{V} and $[\mathcal{C}^{\text{op}}, \mathcal{V}]$. If for all \mathcal{V} -functors $G : \mathcal{D} \odot \mathcal{C} \rightarrow \mathcal{V}$ there is an isomorphism

$$U \star \{X, G\} \cong \{X, U \star G\}$$

then one says that U -weighted colimits commute with X -weighted limits in \mathcal{V} . This is equivalent to saying that the functor $U \star _ : [\mathcal{D}, \mathcal{V}] \rightarrow \mathcal{V}$ preserves X -weighted limits and also to that the functor $\{X, _ \} : [\mathcal{C}^{\text{op}}, \mathcal{V}] \rightarrow \mathcal{V}$ preserves the U -weighted colimits.

For a fixed X let $\mathcal{W}_X \subseteq [\mathcal{C}^{\text{op}}, \mathcal{V}]$ be the full subcategory of weights U such that $U \star _$ preserves X -weighted limits. Then it is evident that \mathcal{W}_X is a self-cocomplete replete full subcategory of $[\mathcal{C}^{\text{op}}, \mathcal{V}]$.

Example 8.6. The category $\text{Flat}(\mathcal{C}^{\text{op}}, \mathcal{V})$ consists of functors F such that the F -weighted colimits commute with all finite limits. Just like in the previous example it is easy to see that $\text{Flat}(\mathcal{C}^{\text{op}}, \mathcal{V}) \subseteq [\mathcal{C}^{\text{op}}, \mathcal{V}]$ is a self-cocomplete replete full subcategory. Similar can be said about the category $\alpha\text{-Flat}(\mathcal{C}^{\text{op}}, \mathcal{V})$ of α -flat functors (a.k.a. α -filtered weights) where α is a regular cardinal. For the precise definitions of these notions we refer to [6] which reveals also that in order for these examples to work we have to assume that \mathcal{V} is a locally α_0 -presentable base and $\alpha \geq \alpha_0$. Many of the usual base categories are in fact locally finitely presentable bases.

Remark 8.7. \mathcal{V} -categories equivalent to self-cocomplete subcategories of presheaves have a certain resemblance to accessible categories [1, 6] after a little reformulation. Let $\mathcal{C} \hookrightarrow \mathcal{M}$ be a full small subcategory. Let $\mathcal{W}' \subseteq [\mathcal{C}^{\text{op}}, \mathcal{V}]$ be any full replete subcategory such that

- \mathcal{M} has \mathcal{W}' -weighted colimits,
- each $C \in \mathcal{C}$ is " \mathcal{W}' -presentable", i.e., $\mathcal{M}(C, _) : \mathcal{M} \rightarrow \mathcal{V}$ preserves \mathcal{W}' -weighted colimits,
- every $M \in \mathcal{M}$ is the \mathcal{W}' -weighted colimit of the inclusion functor $\mathcal{C} \hookrightarrow \mathcal{M}$.

Then there is a full replete self-cocomplete $\mathcal{W} \subseteq [\mathcal{C}^{\text{op}}, \mathcal{V}]$ (in fact $\mathcal{W} \subseteq \mathcal{W}'$) such that $\mathcal{M} \simeq \mathcal{W}$ as extensions of \mathcal{C} .

In the next two examples the small category \mathcal{C} is a one-object \mathcal{V} -category, i.e., a monoid E in \mathcal{V} .

Example 8.8. Assume \mathcal{V} is preadditive, i.e., an \mathbf{Ab} -category. Let $\mathcal{W}_{\text{fgp}} \subseteq \mathcal{V}_E$ be the full subcategory of finitely generated projective right E -modules. Then the right regular E_E belongs to \mathcal{W}_{fgp} . Let $\phi : {}_E\mathcal{V}_E \rightarrow \mathcal{V}_E$ denote the forgetful functor. If $U \in \mathcal{W}_{\text{fgp}}$ and $V \in {}_E\mathcal{V}_E$ is such that $\phi V \in \mathcal{W}_{\text{fgp}}$ then there exist dual bases, i.e., finite diagrams

$$U \xrightarrow{f_i} E_E \xrightarrow{u_i} U \quad \text{such that} \quad \sum_{i=1}^n u_i \circ f_i = U$$

and

$$\phi V \xrightarrow{g_j} E_E \xrightarrow{v_j} \phi V \quad \text{such that} \quad \sum_{j=1}^m v_j \circ g_j = \phi V.$$

Therefore

$$U \underset{E}{\odot} V \xrightarrow{f_i \underset{E}{\odot} V} E \underset{E}{\odot} V \xrightarrow{\sim} \phi V \xrightarrow{g_j} E_E \xrightarrow{v_j} \phi V \xrightarrow{\sim} E \underset{E}{\odot} V \xrightarrow{u_i \underset{E}{\odot} V} U \underset{E}{\odot} V$$

is also a dual basis proving that \mathcal{W}_{fgp} is self-cocomplete.

Example 8.9. Let $\mathcal{W}_1 \subseteq \mathcal{V}_E$ be the full subcategory of rank 1 free right E -modules. If $U \in \mathcal{W}_1$ and $V \in {}_E\mathcal{V}_E$ is such that $\phi V \in \mathcal{W}_1$ then $U \underset{E}{\odot} V \cong E \underset{E}{\odot} V \cong \phi V \cong E_E$. Therefore \mathcal{W}_1 is self-cocomplete.

For $\mathcal{V} = \mathbf{Ab}$ and E a ring, many constructions of ring theory lead also to self-cocomplete subcategories of \mathcal{V}_E such as finitely generated modules, projective modules and finitely presented modules.

Example 8.10. This is an example of a $\mathcal{W} \subseteq \mathbf{Ab}_E$ which is not self-cocomplete. Let E be a field and let \mathcal{W} be the full subcategory of E -vector spaces of dimension less than n where $n > 2$. Let E^2 be the 2-dimensional vector space and define the functor $F : E \rightarrow \mathcal{V}_E$, i.e., the bimodule F , as E^2 with left E -action equal to the right E -action. Then the W -weighted colimit $W \star F$ is just the tensor product $W \otimes E^2$ of vector spaces and has dimension $2 \dim W$. Hence \mathcal{W} satisfies (sc-1) but not (sc-2).

Now let \mathcal{M} be an arbitrary \mathcal{V} -category and \mathcal{C} be a small \mathcal{V} -category. For a \mathcal{V} -functor $J : \mathcal{C} \rightarrow \mathcal{M}$ we define $J_* : \mathcal{M} \rightarrow [\mathcal{C}^{\text{op}}, \mathcal{V}]$ as the \mathcal{V} -functor sending M to the presheaf $\mathcal{M}(J_-, M)$. Let $\mathcal{W} \subseteq [\mathcal{C}^{\text{op}}, \mathcal{V}]$ denote the full replete subcategory of objects $J_* M$, $M \in \mathcal{M}$. Assume that

(wb-0) the \mathcal{W} -weighted colimits of any functor $F : \mathcal{C} \rightarrow \mathcal{M}$ exist in \mathcal{M} .

Then we can introduce (the enriched version of) the skew monoidal structure that Altenkirch, Chapman and Uustalu construct [3] on the functor category $[\mathcal{C}, \mathcal{M}]$. (For the precise connection we note that the left Kan extension of G along J becomes $\text{Lan}_J G = G^* J'_*$ in our notation.)

The skew monoidal product of F and $G \in [\mathcal{C}, \mathcal{M}]$ is a (chosen) \mathcal{W} -weighted pointwise colimit $(F \otimes G)C := J_*(FC) \star G$, i.e., it is the composite functor

$$F \otimes G := \left(\mathcal{C} \xrightarrow{F} \mathcal{M} \xrightarrow{J'_*} \mathcal{W} \xrightarrow{G^*} \mathcal{M} \right)$$

where we introduced $G^* := _ \star G$, the weighted colimit of G as a functor of the weight, and J'_* denotes J_* with codomain restricted to \mathcal{W} . Notice that then J^* is left adjoint to J'_* . Still, when no confusion arises, we write J_* in place of J'_* .

The monoidal unit is the object $J \in [\mathcal{C}, \mathcal{M}]$.

The skew monoidal comparison morphisms are

$$\begin{aligned} \varepsilon_F : F \otimes J &\rightarrow F, \quad (\varepsilon_F)_C = \text{ev}_{FC} : J^* J_* FC \rightarrow FC \\ \eta_G : G &\rightarrow J \otimes G, \quad (\eta_G)_C = \left(GC \xrightarrow{\sim} G^* YC \xrightarrow{G^* J} G^* J_* JC \right) \\ \gamma_{F,G,H} : F \otimes (G \otimes H) &\rightarrow (F \otimes G) \otimes H, \\ &= \left((H^* J_* G)^* J_* F \xrightarrow{\sim} H^* (J_* G)^* J_* F \xrightarrow{H^* J_{J_* F, G}} H^* J_* G^* J_* F \right) \end{aligned}$$

where $\text{ev}_M : \mathcal{M}(J_-, M) \star J \rightarrow M$ is the counit of the adjunction $J^* \dashv J'_*$, the isomorphism in $\gamma_{F,G,H}$ comes from interchangeability of colimits and $J_{U,G} : U \star J_* G \rightarrow J_*(U \star G)$ is the \mathcal{V} -natural transformation

$$\begin{aligned} I &\xrightarrow{i_{U \star G}} \mathcal{M}(U \star G, U \star G) \xrightarrow{\sim} [\mathcal{C}^{\text{op}}, \mathcal{V}](U, \mathcal{M}(G_-, U \star G)) \rightarrow \\ &\xrightarrow{(J_*)_-} [\mathcal{C}^{\text{op}}, \mathcal{V}](U, [\mathcal{C}^{\text{op}}, \mathcal{V}](J_* G_-, J_*(U \star G))) \xrightarrow{\sim} [\mathcal{C}^{\text{op}}, \mathcal{V}](U \star J_* G, J_*(U \star G)) \end{aligned}$$

Invertibility of $J_{U,G}$ means precisely that J_* preserves the U -weighted colimit of G .

The authors of [3] call $J : \mathcal{C} \rightarrow \mathcal{M}$ *well-behaved* if, beyond the condition (wb-0) above, the following three conditions hold.

- (wb-1) $J : \mathcal{C} \rightarrow \mathcal{M}$ is fully faithful.
- (wb-2) $J_* : \mathcal{M} \rightarrow [\mathcal{C}^{\text{op}}, \mathcal{V}]$ is fully faithful.
- (wb-3) J_* preserves the \mathcal{W} -weighted colimits of all $G : \mathcal{C} \rightarrow \mathcal{M}$.

It is then proved in [3, Theorem 4.4] that if J is well-behaved then the skew monoidal category $\langle [\mathcal{C}, \mathcal{M}], J, \otimes, \gamma, \eta, \varepsilon \rangle$ defined above is a monoidal category. In the above presentation this is rather clear: (wb-1) implies that η is invertible, (wb-2) implies that ε is invertible and (wb-3) implies that γ is invertible.

Theorem 8.11. *Let \mathcal{M} be a \mathcal{V} -category and let \mathcal{C} be a small \mathcal{V} -category. Then there exists a well-behaved $J : \mathcal{C} \rightarrow \mathcal{M}$ precisely when \mathcal{M} is equivalent to a self-cocomplete full replete subcategory $\mathcal{W} \subseteq [\mathcal{C}^{\text{op}}, \mathcal{V}]$. In this case the monoidal category $\langle [\mathcal{C}, \mathcal{M}], J, \otimes, \gamma, \eta, \varepsilon \rangle$ is equivalent to the monoidal category $\underline{\mathcal{W}}$ of those bimodules $F : \mathcal{C} \odot \mathcal{C}^{\text{op}} \rightarrow \mathcal{V}$ which as right \mathcal{C} -modules belong to \mathcal{W} .*

Proof. If a well-behaved $J : \mathcal{C} \rightarrow \mathcal{M}$ exists then the fully faithful J_* , with codomain restricted to its replete image \mathcal{W} , defines an equivalence $\mathcal{M} \simeq \mathcal{W}$. By (wb-1) $Y \cong J_* J$ therefore \mathcal{W} satisfies axiom (sc-1) of self-cocompleteness. Since J_* preserves the \mathcal{W} -weighted colimits by (wb-3), all of which exists in \mathcal{M} by (wb-0), the \mathcal{W} satisfies (sc-2). Therefore \mathcal{W} is self-cocomplete.

Assume that $\Psi : \mathcal{M} \xrightarrow{\sim} \mathcal{W}$ is an equivalence with a self-cocomplete full replete $\mathcal{W} \subseteq [\mathcal{C}^{\text{op}}, \mathcal{V}]$. Then Ψ is part of an adjoint equivalence $\Psi^* \dashv \Psi$. Define $J := \Psi^* Y_{\mathcal{W}} : \mathcal{C} \rightarrow \mathcal{M}$ which is fully faithful obviously, so (wb-1) holds. Then the associated $J_* = \mathcal{M}(J, -)$ satisfies

$$\begin{aligned} J_* M &= \mathcal{M}(\Psi^* Y_{\mathcal{W}} -, M) \cong \mathcal{W}(Y_{\mathcal{W}} -, \Psi M) = \\ &= [\mathcal{C}^{\text{op}}, \mathcal{V}](Y, \Psi M) \cong \Psi M \end{aligned}$$

by Yoneda Lemma therefore J_* has replete image precisely \mathcal{W} . Moreover the corestriction $J'_* : \mathcal{M} \rightarrow \mathcal{W}$ is isomorphic to Ψ implying that J_* is fully faithful, so (wb-2) holds for J . The colimit of $F : \mathcal{C} \rightarrow \mathcal{M}$ weighted by $U \in \mathcal{W}$ can now be constructed from its existence in \mathcal{W} ,

$$U \star F := \Psi^*(U \star \Psi F).$$

Indeed,

$$\begin{aligned}\mathcal{M}(U \star F, M) &\cong \mathcal{W}(U \star \Psi F, \Psi M) \cong [\mathcal{C}^{\text{op}}, \mathcal{V}](U, \mathcal{W}((\Psi F)_-, \Psi M)) \cong \\ &\cong [\mathcal{C}^{\text{op}}, \mathcal{V}](U, \mathcal{M}(F_-, M)).\end{aligned}$$

This proves (wb-0). It remains to show (wb-3). First notice that the inclusion $\mathcal{W} \subseteq [\mathcal{C}^{\text{op}}, \mathcal{V}]$ both preserves and reflects \mathcal{W} -weighted colimits. Therefore J_* preserves the \mathcal{W} -weighted colimits iff J'_* does. But J'_* is isomorphic to the equivalence $\Psi : \mathcal{M} \xrightarrow{\sim} \mathcal{W}$ so it preserves every colimit that exist in \mathcal{M} . This proves (wb-3) hence also the first statement. As for the second, let $J : \mathcal{C} \rightarrow \mathcal{M}$ be well-behaved and let \mathcal{W} be the self-cocomplete replete image of J_* . Then composing functors with J_* defines a functor

$$[\mathcal{C}, \mathcal{M}] \rightarrow [\mathcal{C}, \mathcal{W}] \equiv \underline{\mathcal{W}}, \quad F \mapsto J'_*F$$

which is an equivalence since J'_* is. But it is also a strong monoidal functor because $J'_*J \cong Y_{\mathcal{W}}$ and

$$J_*F \otimes J_*G = \int^{\mathcal{C}} \mathcal{M}(JC, F_-) \odot \mathcal{M}(J_-, GC) \cong \mathcal{M}(J_-, \int^{\mathcal{C}} \mathcal{M}(JC, F_-) \odot GC) = J_*(F \otimes G)$$

due to property (wb-3). \square

The above Theorem is worth a comparison with the general theory of free cocompletions [12, 2] which offers an elegant explanation for the monoidal structure on $[\mathcal{C}, \mathcal{M}]$. Let $\Phi = \{\phi_i : \mathcal{I}_i \rightarrow \mathcal{V} \mid i \in \mathcal{I}\}$ be a family of small weights. Then for a small category \mathcal{C} there is the closure $\Phi(\mathcal{C}) \subseteq [\mathcal{C}^{\text{op}}, \mathcal{V}]$ of \mathcal{C} under Φ -colimits. Let $\Phi\text{-Cocts}$ denote the 2-category of Φ -cocomplete \mathcal{V} -categories, Φ -cocontinuous functors and all natural transformations between them. Then composition with the inclusion functor $J : \mathcal{C} \rightarrow \Phi(\mathcal{C})$ defines an equivalence [2, Prop. 4.1.]

$$\Phi\text{-Cocts}(\Phi(\mathcal{C}), \mathcal{N}) \xrightarrow{\sim} [\mathcal{C}, \mathcal{N}]$$

for any Φ -cocomplete \mathcal{N} . Inserting $\mathcal{N} = \Phi(\mathcal{C})$ and using the fact that the endohom category of any 2-category is a monoidal category the above equivalence induces a monoidal structure on $[\mathcal{C}, \Phi(\mathcal{C})]$. By [2, Prop. 4.1.] or [12, Theorem 5.35] we know that the inverse equivalence is given by left Kan extension along J . Therefore it is not difficult to see that the induced monoidal structure of $[\mathcal{C}, \Phi(\mathcal{C})]$ is precisely the skew monoidal structure of [3] with a well-behaved J .

Since $\text{Lan}_J F$ of a functor $F \in [\mathcal{C}, \Phi(\mathcal{C})]$ is given pointwise by a colimit

$$\text{Lan}_J F(M) = J_*M \star F, \quad M \in \Phi(\mathcal{C})$$

the weight J_*M of which belongs to $\Phi(\mathcal{C})$, the very existence of $\text{Lan}_J F$ shows that $\Phi(\mathcal{C})$ is self-cocomplete. Vice versa, if $\mathcal{W} \subseteq [\mathcal{C}^{\text{op}}, \mathcal{V}]$ is self-cocomplete then, obviously, \mathcal{W} is the Φ -cocompletion of \mathcal{C} with $\Phi = \mathcal{W}$. Thus being self-cocomplete is a property that detects whether a full replete subcategory of $[\mathcal{C}^{\text{op}}, \mathcal{V}]$ is the Φ -cocompletion of \mathcal{C} w.r.t. some family Φ of weights.

Applying the above notions and results to our problem of monoidality of the module category of a skew monoidal tensored \mathcal{V} -category we meet only very special self-cocomplete subcategories \mathcal{W} of $[E^{\text{op}}, \mathcal{V}] = \mathcal{V}_E$, namely reflective ones. But even for a cocomplete \mathcal{W} the condition of self-cocompleteness is non-trivial since the inclusion $\mathcal{W} \hookrightarrow \mathcal{V}_E$ need not preserve all (small) colimits.

Proposition 8.12. *Let \mathcal{M} be a tensored \mathcal{V} -category with reflexive coequalizers and with a dense generator R . Let $E = \mathcal{M}(R, R)$ and let the category \mathcal{V}_E be endowed with the skew monoidal product $V \odot' W := \phi'V \odot W$, where $\phi' : {}_E\mathcal{V}_E \rightarrow \mathcal{V}_E$ is the forgetful functor.*

- (i) \mathcal{M} is a reflective subcategory of the presheaf category \mathcal{V}_E , hence complete and cocomplete.
- (ii) The inclusion functor $J : \mathcal{M} \hookrightarrow \mathcal{V}_E$, $M \mapsto \mathcal{M}(\underline{R}, M)$, exhibits $\langle \mathcal{M}, \bullet, R \rangle$ as the skew monoidal reflection of $\langle \mathcal{V}_E, \odot', E' \rangle$ in the sense of [15, Theorem 2.1].

- (iii) The lift of $J : \mathcal{M} \hookrightarrow \mathcal{V}_E$ to the Eilenberg-Moore categories is a functor $\mathcal{J} : {}_E\mathcal{M} \rightarrow {}_E\mathcal{V}_E$ which exhibits $\langle {}_E\mathcal{M}, \underline{\otimes}, \underline{R} \rangle$ as the skew monoidal reflection of the ordinary monoidal \mathcal{V} -category $\langle {}_E\mathcal{V}_E, \odot, E \rangle$.
- (iv) If $\mathcal{J} : {}_E\mathcal{M} \rightarrow {}_E\mathcal{V}_E$ is strong monoidal then $\langle {}_E\mathcal{M}, \underline{\otimes}, \underline{R} \rangle$ is a closed monoidal reflective subcategory of the bimodule category $\langle {}_E\mathcal{V}_E, \odot, E \rangle$ and \mathcal{J} is strong closed.
- (v) If the replete image of $J : \mathcal{M} \hookrightarrow \mathcal{V}_E$ is a self-cocomplete subcategory of \mathcal{V}_E then the functor $\mathcal{J} : {}_E\mathcal{M} \rightarrow {}_E\mathcal{V}_E$ is strong monoidal.

Proof. (i) Existence of tensors and reflexive coequalizers in \mathcal{M} imply the existence of the colimits $V \odot \underline{R}$ for all $V \in \mathcal{V}_E$ and this defines a left adjoint J^* of J .

(ii) The condition in [15, Theorem 2.1] for $J^* \dashv J$ to be a (right) skew monoidal reflection is invertibility of the arrows $J^*(JM \odot' \text{coev}_{\underline{R}, V})$ for $M \in \mathcal{M}$, $V \in \mathcal{V}_E$, which holds automatically due to invertibility of $\text{coev}_{\underline{R}, V} \odot \underline{R}$. The skew monoidal product induced by the reflection is isomorphic to \bullet since

$$M \bullet N = HM \odot N \cong HM \odot (JN \odot \underline{R}) \cong J^*(\phi' JM \odot JN) = J^*(JM \odot' JN).$$

(iii) It is easy to verify that the Eilenberg-Moore category of the canonical monad for $\langle \mathcal{V}_E, \odot', E' \rangle$ is precisely $\langle {}_E\mathcal{V}_E, \odot, E \rangle$. The lift of $J : \mathcal{M} \hookrightarrow \mathcal{V}_E$ is the functor $\mathcal{J} : {}_E\mathcal{M} \hookrightarrow {}_E\mathcal{V}_E$ of composition with J and it has a left adjoint $\mathcal{J}^* \underline{V} = \underline{V} \odot \underline{R}$. The condition for $\langle {}_E\mathcal{M}, \underline{\otimes}, \underline{R} \rangle$ to arise from $\langle {}_E\mathcal{V}_E, \odot, E \rangle$ as a skew reflection is invertibility of $\mathcal{J}^*(\mathcal{J}\underline{M} \odot \text{coev}_{\underline{R}, \underline{V}})$ which holds automatically as in (ii). Then the isomorphism $\underline{M} \otimes \underline{N} \cong \mathcal{J}^*(\mathcal{J}\underline{M} \odot \mathcal{J}\underline{N})$ can be verified easily.

(iv) Conservativity of the fully faithful \mathcal{J} implies that $\langle {}_E\mathcal{M}, \underline{\otimes}, \underline{R} \rangle$ is a monoidal category. Since $\langle {}_E\mathcal{V}_E, \odot, E \rangle$ is closed, we are in the situation of a closed reflection [7] and the adjunction $\mathcal{J}^* \dashv \mathcal{J}$ induces left and right internal homs on $\langle {}_E\mathcal{M}, \underline{\otimes}, \underline{R} \rangle$ and \mathcal{J} preserves them.

(v) The skew monoidal structure of \mathcal{J} defined by the reflection consists of the identity $E = \mathcal{J}\underline{R}$ and of the arrows $\mathcal{J}_{M, N} := \text{coev}_{\underline{R}, \mathcal{J}\underline{M} \odot \mathcal{J}\underline{N}}$. By assumption the J preserves the JM -weighted colimits and this implies in particular that the strengths $\mathcal{J}_{\underline{M}, \underline{N}}$ are invertible. Using (92) this means that \mathcal{J} is strong monoidal. \square

In the Theorem below we say that an object $R \in \mathcal{M}$ well-generates \mathcal{M} if the corresponding \mathcal{V} -functor $E \rightarrow \mathcal{M}$ is well-behaved.

Theorem 8.13. *Let $\langle \mathcal{M}, *, R \rangle$ be an r -exact skew monoidal tensored \mathcal{V} -category in which \mathcal{M} is well-generated by R . Let E be the endomorphism monoid of R . Then*

- (i) the \mathcal{V} -category \mathcal{M} is complete and cocomplete and is equivalent to a self-cocomplete full replete and reflective subcategory $\mathcal{W} \subseteq \mathcal{V}_E$,
- (ii) $\langle {}_E\mathcal{M}, \underline{\otimes}, \underline{R} \rangle$ is a closed monoidal \mathcal{V} -category monoidally equivalent to the subcategory $\underline{\mathcal{W}} \subseteq {}_E\mathcal{V}_E$ of those bimodules which, as right E -modules, belong to \mathcal{W} .
- (iii) the skew monoidal category $\langle \mathcal{M}^T, \bar{\otimes}, \bar{R} \rangle$ of modules over \mathcal{M} is a monoidal category and the forgetful functor $\mathcal{G} : \mathcal{M}^T \rightarrow {}_E\mathcal{M}$ is strong monoidal and monadic,
- (iv) the monoidal category $\langle \mathcal{M}^T, \bar{\otimes}, \bar{R} \rangle$ is equivalent to the Eilenberg-Moore category of an opmonoidal monad defined on $\underline{\mathcal{W}}$.

Proof. (i) follows from Proposition 8.12 (i) and Theorem 8.11.

(ii) By Proposition 8.12 (v) and (iv) $\langle {}_E\mathcal{M}, \underline{\otimes}, \underline{R} \rangle$ is closed monoidal and using (i) ${}_E\mathcal{M} = [E, \mathcal{M}]$ is equivalent to the monoidal category $[E, \mathcal{W}] = \underline{\mathcal{W}}$ by Corollary 8.4.

(iii) Using that well-generators are dense generators we obtain from Theorem 7.5 that \mathcal{G} is strong skew monoidal and monadic. Since the codomain of \mathcal{G} is monoidal and \mathcal{G} reflects isomorphisms, the skew monoidal functor relations imply that the domain of \mathcal{G} is also a monoidal category.

(iv) The equivalence of \mathcal{M}^T with $\underline{\mathcal{W}}^T$ for an opmonoidal monad follows also from Theorem 7.5 because ${}_E\mathcal{M}$ is equivalent to $\underline{\mathcal{W}}$. \square

APPENDIX A. ENRICHED E -OBJECTS

In [19] we have shown for $\mathcal{V} = \mathbf{Set}$ that any skew monoidal structure $\langle \mathcal{M}, *, R \rangle$ on \mathcal{M} induces a skew monoidal structure $\langle {}_E\mathcal{M}, *_q, \underline{R} \rangle$ on the category of E -objects. This Appendix serves as a supplement to [19, Theorem 5.3] in which the equivalence of \mathcal{M}^T with $({}_E\mathcal{M})^{T_q}$ is proven merely as a category equivalence. Here we show, using the Lifting Theorem, that this is actually a skew monoidal equivalence and this holds for general \mathcal{V} . Also we need to reformulate some Lemmas of [19] in order to adapt them to the enriched setting.

Let $\langle \langle \mathcal{M}, c, i \rangle, \langle \odot, a, l \rangle, \langle *, \Gamma, \Gamma' \rangle, R, \gamma, \eta, \varepsilon \rangle$ be a skew monoidal tensored \mathcal{V} -category.

Let $E := \langle \mathcal{M}(R, R), c_{R,R,R}, i_R \rangle$ be the endomorphism monoid of the skew unit object R . The algebras for the monad $E \odot _$ will be called E -objects. So, the E -objects are pairs $\langle M, \lambda \rangle$ where M is an object of \mathcal{M} and $\lambda : E \odot M \rightarrow M$ is an arrow satisfying

$$(103) \quad \lambda \circ (E \odot \lambda) = \lambda \circ (c_{R,R,R} \odot M) \circ a_{E,E,M}$$

$$(104) \quad \lambda \circ (i_R \odot M) \circ l_M = M.$$

A morphism $\langle M, \lambda \rangle \rightarrow \langle M', \lambda' \rangle$ of E -objects is an arrow $t : M \rightarrow M'$ satisfying $\lambda' \circ (E \odot t) = t \circ \lambda$.

The category of E -objects ${}_E\mathcal{M}$ can alternatively be defined as the category $[E, \mathcal{M}]$ of \mathcal{V} -functors from the 1-object \mathcal{V} -category E to \mathcal{M} . The equivalence of the two definitions is provided by the adjunction relations

$$(105) \quad \varphi = \mathcal{M}(M, \lambda) \circ \text{coev}_{M,E}$$

$$(106) \quad \lambda = \text{ev}_{M,M} \circ (\varphi \odot M)$$

between the \mathcal{V} -functor data $\varphi : E \rightarrow \mathcal{M}(M, M)$ and the action data $\lambda : E \odot M \rightarrow M$.

Using the symmetry s in \mathcal{V} we can define the monoid $E^{\text{op}} := \langle \mathcal{M}(R, R), c_{R,R,R}^{\text{op}}, i_R \rangle$, with opposite multiplication $c_{R,R,R}^{\text{op}} = c_{R,R,R} \circ s_{E,E}$, and E^{op} -objects as pairs $\langle M, \rho \rangle$ where $\rho : E \odot M \rightarrow M$ is the same type of arrow as in an E -object but satisfies different axioms, namely (103) and (104) with c replaced by c^{op} . In other words, E^{op} -objects are \mathcal{V} -functors $E^{\text{op}} \rightarrow \mathcal{M}$.

U -objects in \mathcal{M} for an arbitrary monoid $\langle U, u_2, u_0 \rangle$ in \mathcal{V} can be defined similarly. If U and V are monoids in \mathcal{V} then $W = U \odot V$ is also a monoid with multiplication w_2 which is, up to invertible associators, the composite $(u_2 \odot v_2) \circ (U \odot s_{V,U} \odot V)$. Given a U -object $\langle M, \kappa \rangle$ and a V -object $\langle M, \lambda \rangle$ on the same underlying M the necessary and sufficient condition for the existence of a $U \odot V$ -object $(U \odot V) \odot M \rightarrow M$ extending both κ and λ is that " κ and λ commute" in the sense of obeying

$$\kappa \circ (U \odot \lambda) = \lambda \circ (V \odot \kappa) \circ a_{V,U,M}^{-1} \circ (s_{U,V} \odot M) \circ a_{U,V,M}.$$

The following Lemma is the enriched version of [19, Lemma 4.1] in which we use the definitions

$$\mu_{M,N} := (\varepsilon_M * N) \circ \gamma_{M,R,N}$$

$$\sigma_N := (\text{ev}_{R,R} * N) \circ \Gamma_{E,R,N} \circ (E \odot \eta_N)$$

and remark that $\sigma : E \odot _ \rightarrow R * _$ is a monad morphism [18],

$$(107) \quad \mu_N \circ \sigma_{R*N} \circ (E \odot \sigma_N) = \sigma_N \circ (c_{R,R,R} \odot N) \circ a_{E,E,N}$$

$$(108) \quad \sigma_N \circ (i_R \odot N) \circ l_N = \eta_N,$$

and μ satisfies

$$(109) \quad \mu_{M,N} \circ \mu_{M,R*N} = \mu_{M,N} \circ (M * \mu_N)$$

$$(110) \quad \mu_{M,N} \circ (M * \eta_N) = M * N.$$

Lemma A.1. *Let M and N be objects in a skew monoidal tensored \mathcal{V} -category \mathcal{M} .*

(i) *Then $\langle M * N, \rho \rangle$ is an E^{op} -object where*

$$\rho := \mu_{M,N} \circ (M * \sigma_N) \circ \Gamma'_{E,M,N} : E \odot (M * N) \rightarrow M * N.$$

(ii) *If $\langle M, \lambda_M \rangle$ is an E -object then so is $\langle M * N, \lambda_1 \rangle$, where*

$$\lambda_1 := (\lambda_M * N) \circ \Gamma_{E,M,N} : E \odot (M * N) \rightarrow M * N,$$

and λ_1 and ρ commute.

(iii) *If $\langle N, \lambda_N \rangle$ is an E -object then so is $\langle M * N, \lambda_2 \rangle$, where*

$$\lambda_2 := (M * \lambda_N) \circ \Gamma'_{E,M,N} : E \odot (M * N) \rightarrow M * N,$$

and λ_2 and ρ commute.

(iv) *If $\langle M, \lambda_M \rangle$ and $\langle N, \lambda_N \rangle$ are E -objects then the above defined λ_1 and λ_2 also commute.*

If we have three objects then there are two $*$ signs and there are two E^{op} -actions on $L * (M * N)$

$$(111) \quad \rho_1 := \rho_{L,M*N}$$

$$(112) \quad \rho_2 := (L * \rho_{M,N}) \circ \Gamma'_{E,L,M*N}$$

and two E^{op} -actions on $(L * M) * N$

$$(113) \quad \rho'_1 := (\rho_{L,M} * N) \circ \Gamma_{E,L*M,N}$$

$$(114) \quad \rho'_2 := \rho_{L*M,N}.$$

If the three objects are E -objects then there are three E -actions on $L * (M * N)$

$$(115) \quad \lambda_1 := (\lambda_L * (M * N)) \circ \Gamma_{E,L,M*N}$$

$$(116) \quad \lambda_2 := (L * (\lambda_M * N)) \circ (L * \Gamma_{E,M,N}) \circ \Gamma'_{E,L,M*N}$$

$$(117) \quad \lambda_3 := (L * (M * \lambda_N)) \circ (L * \Gamma'_{E,M,N}) \circ \Gamma'_{E,L,M*N}$$

and three E -actions on $(L * M) * N$

$$(118) \quad \lambda'_1 := ((\lambda_L * M) * N) \circ (\Gamma_{E,L,M} * N) \circ \Gamma_{E,L*M,N}$$

$$(119) \quad \lambda'_2 := ((L * \lambda_M) * N) \circ (\Gamma'_{E,L,M} * N) \circ \Gamma_{E,L*M,N}$$

$$(120) \quad \lambda'_3 := ((L * M) * \lambda_N) \circ \Gamma'_{E,L*M,N}.$$

Lemma A.2. *For E -objects \underline{L} , \underline{M} and \underline{N} the skew monoidal structure maps obey the following identities.*

$$(121) \quad \gamma_{L,M,N} \circ \rho_i = \rho'_i \circ (E \odot \gamma_{L,M,N}) \quad (i = 1, 2)$$

$$(122) \quad \gamma_{L,M,N} \circ \lambda_i = \lambda'_i \circ (E \odot \gamma_{L,M,N}) \quad (i = 1, 2, 3)$$

$$(123) \quad \rho_{R,M} \circ (E \odot \eta_M) = \lambda_1 \circ (E \odot \eta_M)$$

$$(124) \quad \lambda_2 \circ (E \odot \eta_M) = \eta_M \circ \lambda_M$$

$$(125) \quad \varepsilon_M \circ \rho_{M,R} = \varepsilon \circ \lambda_2$$

$$(126) \quad \varepsilon_M \circ \lambda_1 = \lambda_M \circ (E \odot \varepsilon_M)$$

where the unit object R is considered as an E -object via $\lambda_R = \text{ev}_{R,R} : E \odot R \rightarrow R$.

Proof. Relations (122), (124), (126) are immediate consequences of the skew monoidality axioms (34)-(38) that express the fact that γ , η , ε are transformations of category morphisms. The remaining relations are also not difficult. \square

Now assume that \mathcal{M} is r2-exact and define the skew monoidal product $*_q$ of two E -objects by the reflexive coequalizer

$$(127) \quad E \odot (M * N) \underset{\lambda_2}{\overset{\rho}{\rightrightarrows}} M * N \xrightarrow{q_{M,N}} M *_q N$$

in \mathcal{M} which can then be lifted to the E -object $\underline{M} *_q \underline{N} = \langle M *_q N, \lambda_1 \rangle$. The unit object for $*_q$ is the E -object $\underline{R} = \langle R, \text{ev}_{R,R} \rangle$.

Proposition A.3. *For any r2-exact skew monoidal tensored \mathcal{V} -category $\langle \mathcal{M}, *, R \rangle$ the category ${}_E\mathcal{M}$ of E -objects in \mathcal{M} has a unique skew monoidal tensored \mathcal{V} -category structure with skew monoidal product $*_q$ and with skew monoidal unit \underline{R} such that the forgetful functor $\phi : {}_E\mathcal{M} \rightarrow \mathcal{M}$ together with the natural transformation $q_{M,N}$ of (127) and with the identity arrow $R \rightarrow \phi \underline{R}$ is a skew monoidal functor. The resulting skew monoidal tensored \mathcal{V} -category $\langle {}_E\mathcal{M}, *_q, \underline{R} \rangle$ is r2-exact.*

Proof. On the level of ordinary categories and functors the unique skew monoidal structure $\langle {}_E\mathcal{M}, *_q, \underline{R}, \gamma^q, \eta^q, \varepsilon^q \rangle$ that makes ϕ skew monoidal has already been constructed in [19, Proposition 4.3]. Although the right exactness conditions of [19, Proposition 4.3] were somewhat too generous, by inspecting the proof one can see that what is really needed is only the existence of reflexive coequalizers and preservation of them by all the endofunctors $M * _$. So, it suffices here to deal with the \mathcal{V} -actegory structure.

Like for \mathcal{M}^T in the proof of Lemma 3.3, the action of \mathcal{V} on ${}_E\mathcal{M}$ is defined by

$$V \odot \underline{M} := \langle V \odot M, (V \odot \lambda_M) \circ E_{V,M}^{-1} \rangle$$

and the \mathcal{V} -valued hom by the equalizer

$${}_E\mathcal{M}(\underline{M}, \underline{N}) \longrightarrow \mathcal{M}(M, N) \xrightarrow[\mathcal{M}(\lambda_{M,N})]{\mathcal{M}(E \odot M, \lambda_N) \circ (E \odot _)_{M,N}} \mathcal{M}(E \odot M, N)$$

where the \mathcal{V} -functor structure map $(E \odot _)_{M,N} : \mathcal{M}(M, N) \rightarrow \mathcal{M}(E \odot M, E \odot N)$ is constructed using (22) from the strength

$$(128) \quad E_{V,M} := a_{E,V,M}^{-1} \circ (s_{V,E} \odot M) \circ a_{E,V,M} : V \odot (E \odot M) \rightarrow E \odot (V \odot M)$$

of the (strong) morphism $E \odot _ : \mathcal{M} \rightarrow \mathcal{M}$ of \mathcal{V} -actegories. It is then easy to check that $_ \odot \underline{M}$ is left adjoint to ${}_E\mathcal{M}(\underline{M}, _)$, so that ${}_E\mathcal{M}$ is a hommed \mathcal{V} -actegory.

The two strengths of the 2-variable \mathcal{V} -actegory morphism $*_q$ are uniquely determined by those of $*$ due to the requirement that $q_{M,N}$ be a transformation of category morphisms:

$$(129) \quad \begin{array}{ccc} V \odot (M * N) & \xrightarrow{V \odot q_{M,N}} & V \odot (M *_q N) \\ \Gamma'_{V,M,N} \downarrow & & \downarrow \Gamma'_{V,M,N} = \phi \Gamma'_{V,\underline{M},\underline{N}}^q \\ M * (V \odot N) & \xrightarrow{q_{M,V \odot N}} & M *_q (V \odot \underline{N}) \end{array}$$

and a similar diagram defines $\Gamma_{V,\underline{M},\underline{N}}^q : V \odot (\underline{M} *_q \underline{N}) \rightarrow (V \odot \underline{M}) *_q \underline{N}$.

The coherence conditions that γ^q , η^q and ε^q should satisfy in order that they become transformations of category morphisms follow from the analogous coherence conditions (34–38) for γ , η and ε by using that $q_{M,N}$ is a coequalizer, $V \odot _$ preserves such coequalizers and ϕ is faithful.

Since \mathcal{M} is r2-exact, the monad $E \odot _$ preserves reflexive coequalizers therefore its Eilenberg-Moore category ${}_E\mathcal{M}$ also has reflexive coequalizers. Furthermore, $\underline{M} *_q _$ preserves such coequalizers

because both functors $M *_q \phi _$ and $E \odot (M *_q \phi _)$ occurring in (127) preserve them. The strength Γ'^q in (129) is an isomorphism since both horizontal arrows are coequalizers and Γ' is an isomorphism. Finally, $V \odot _ : {}_E\mathcal{M} \rightarrow {}_E\mathcal{M}$ also preserves reflexive coequalizers because $V \odot _ : \mathcal{M} \rightarrow \mathcal{M}$ does and ϕ creates reflexive coequalizers. This proves r2-exactness of $\langle {}_E\mathcal{M}, *_q, \underline{R} \rangle$. \square

Proof of Theorem 5.2 Applying the construction of Section 4 there is an r2-exact skew monoidal tensored \mathcal{V} -category $({}_E\mathcal{M})^{T_q}$ of $T_q = \underline{R} *_q _$ -algebras on ${}_E\mathcal{M}$. Applying also the lifting procedure of Theorem 5.1 to the skew monoidal functor $\phi : {}_E\mathcal{M} \rightarrow \mathcal{M}$ we obtain the $\bar{\phi}$ of Theorem 5.2. By [19, Theorem 5.3. (ii)] this $\bar{\phi}$ is an equivalence of categories therefore it suffices to show strong skew monoidality of $\bar{\phi}$. The $\bar{\phi}_0$ is the identity and $\bar{\phi}_2$ is defined, as in Theorem 5.1, by the diagram

$$(130) \quad \begin{array}{ccccc} M * TN & \xrightarrow[\quad M * (\phi\beta \circ q_{\underline{N}}) \quad]{\mu_{M,N}} & M * N & \xrightarrow{\pi_{M, \bar{\phi}\bar{N}^q}} & M \bar{\otimes} \bar{\phi}\bar{N}^q \\ \downarrow q_{M, T^q \underline{N}} \circ (M * q_{\underline{N}}) & & \downarrow q_{M, \underline{N}} & & \downarrow \bar{\phi}_{M, \bar{N}^q} \\ M *_q T^q \underline{N} & \xrightarrow[\quad M *_q \beta \quad]{\mu_{M, \underline{N}}^q} & M *_q \underline{N} & \xrightarrow{\pi_{M, \bar{N}^q}^q} & M \bar{*}_q \bar{N}^q \end{array}$$

where $\bar{N}^q = \langle \underline{N}, \beta \rangle$ is a T^q -algebra with underlying E -object \underline{N} and with action $\beta : R *_q \underline{N} \rightarrow \underline{N}$, M is an object of \mathcal{M} , $q_{\underline{N}} = q_{R, \underline{N}}$ and $\bar{*}_q$ denotes the horizontal tensor product in $({}_E\mathcal{M})^{T_q}$. The E -object $\underline{N} = \langle N, \lambda_{\underline{N}} \rangle$ is related to the T -algebra $\bar{N} = \bar{\phi}\bar{N}^q$ by the monad morphism σ_N because

$$\begin{aligned} \phi\beta \circ q_{\underline{N}} \circ \sigma_N &= \phi\beta \circ q_{\underline{N}} \circ \lambda_1[R * N] \circ (E \odot \eta_N) = \\ &= \phi\beta \circ \lambda_{R *_q \underline{N}} \circ (E \odot q_{\underline{N}}) \circ (E \odot \eta_N) = \\ &= \lambda_{\underline{N}} \circ (E \odot \phi\beta) \circ (E \odot \eta_N^q) = \lambda_{\underline{N}}. \end{aligned}$$

Therefore composing the parallel pair of the 1st row in (130) with $M * \sigma_N$ we see that $\pi_{M, \bar{N}} \circ \rho = \pi_{M, \bar{N}} \circ \lambda_2$ therefore $\pi_{M, \bar{N}}$ factorizes through $q_{M, \underline{N}}$ by a unique $u : M *_q \underline{N} \rightarrow M \bar{\otimes} \bar{N}$. Then also u factors uniquely through π_{M, \bar{N}^q}^q and this defines an arrow which is necessarily the inverse of $\bar{\phi}_{M, \bar{N}^q}$.

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